Math 520A

Written Assignment No. 5

due Monday, May 2, 2005

Directions. It is intended that you work these as exercises. Although you may refer to books for definitions and standard theorems, searching for solutions to these written exercises either in books or in online references should not be required and is undesirable. If you make use of a reference other than class notes, you must properly cite that use.

You may not seek help from others.

---

1. If F is a field, a polynomial $f(t) \in F[t]$ with coefficients in F determines a "polynomial function" $\varphi(f)$ from F to itself that is defined by

$$(\varphi(f))(a) \;=\; f(a) \text{ for } a \in F \quad .$$

   If A denotes the F-algebra of all functions $F \to F$, φ is an F-algebra homomorphism $F[t] \to A$. Show the following:

   (a) φ is injective if F is an infinite field.

   (b) φ is not injective if F is a finite field.

   (c) φ is not surjective if F is an infinite field.

   (d) φ is surjective if F is a finite field.

2. Recall that a primitive element for a field E as an extension field of F is an element $\theta \in E$ such that $E = F(\theta)$. Find primitive elements for E over $\mathbf{Q}$ in the following cases:

   (a) E is the splitting field over $\mathbf{Q}$ of $t^{12} - 1$.

   (b) $E = \mathbf{Q}(\sqrt{2}, \sqrt{3})$.

3. Show that $t^4 + 1$ is irreducible over the field $\mathbf{Q}$, but, for every prime p, not irreducible over $\mathbf{Z}/p\mathbf{Z}$. Then find the group of $\mathbf{Q}$-algebra automorphisms of the field

$$\mathbf{Q}[t]/(t^4+1)\mathbf{Q}[t] \quad .$$

4. For each of the following irreducible polynomials of degree 3 with coefficients in $\mathbf{Q}$ determine the group of $\mathbf{Q}$-algebra automorphisms of its splitting field:

   (a) $t^3 - 4$.

   (b) $t^3 - 4t + 2$.

   (c) $t^3 - 3t - 1$.

5. For each of the following irreducible polynomials of degree 4 with coefficients in $\mathbf{Q}$ determine the group of $\mathbf{Q}$-algebra automorphisms of its splitting field:

   (a) $t^4 - 10t^2 + 1$.

   (b) $t^4 - 2t^2 - 1$.

   (c) $t^4 - 4t^2 + 2$.

   (d) $t^4 + t - 1$.

   (e) $t^4 + 8t + 12$.