Math 520A
Written Assignment No. 4

due Wednesday, April 13, 2005

Directions. It is intended that you work these as exercises. Although you may refer to books for definitions and standard theorems, searching for solutions to these written exercises either in books or in online references should not be required and is undesirable. If you make use of a reference other than class notes, you must properly cite that use.

You may not seek help from others.

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1. Decompose the polynomial $t^{12} - 1 \in F[t]$ as the product of irreducible polynomials when F is the field
   (a) $\mathbf{Q}$.
   (b) $\mathbf{Z}/5\mathbf{Z}$.

2. In view of the fact that, up to $\mathbf{R}$-algebra isomorphism, the only non-trivial finite extension of the field $\mathbf{R}$ is $\mathbf{C}$, find the number of subfields of the $\mathbf{R}$-algebra
$$\mathbf{R}[t]/(t^4+1)\mathbf{R}[t]$$
containing $\pi(\mathbf{R})$, where
$$\pi : \mathbf{R} \to \mathbf{R}[t]/(t^4+1)\mathbf{R}[t]$$
is the quotient homomorphism, that are isomorphic as $\mathbf{R}$-algebras to $\mathbf{C}$.

3. Recall that the multiplicative group of a finite field must be cyclic. For the irreducible polynomial $p(t) \in F[t]$ find a polynomial in $F[t]$ of degree 1 whose congruence class mod $p(t)$ determines a generator for the multiplicative group of the finite field $F[t]/(p(t))F[t]$ when
   (a) $F = \mathbf{Z}/2\mathbf{Z}, \quad p(t) = t^4 + t + 1.$
   (b) $F = \mathbf{Z}/3\mathbf{Z}, \quad p(t) = t^2 + 1.$
   (c) $F = \mathbf{Z}/3\mathbf{Z}, \quad p(t) = t^3 - t - 1.$
   (d) $F = \mathbf{Z}/2\mathbf{Z}, \quad p(t) = t^5 + t^2 + 1.$

4. Find a monic polynomial $q(t)$ of degree 4 with integer coefficients having
$$\begin{aligned} \alpha &= \sqrt{2} + \sqrt{3} + \sqrt{6} \\ \beta &= -\sqrt{2} + \sqrt{3} - \sqrt{6} \\ \gamma &= \sqrt{2} - \sqrt{3} - \sqrt{6} \\ \delta &= -\sqrt{2} - \sqrt{3} + \sqrt{6} \end{aligned}$$
as real roots. Explain why $q(t)$ must be irreducible.

5. For each of the following monic polynomials p of degree 4 with coefficients in $\mathbf{Q}$ determine the extension degree over $\mathbf{Q}$ of the smallest subfield of $\mathbf{C}$ in which all complex roots of p lie:
   (a) $t^4 - 10t^3 + 35t^2 - 50t + 24.$
   (b) $t^4 + 2.$
   (c) $t^4 - 2t^2 - 1.$
   (d) $t^4 + t - 1.$
   (e) $t^4 - 4t^2 + 2.$