MAT 424/524
Midterm WarmUp

Informal Exercises

October 16, 2002

1. Let V be the vector space of all polynomials of degree at most 3 with coefficients in the field F. Let $\varphi \in \operatorname{End}(V)$ be defined for each $f \in V$ by
$$\phi(f(t)) \;=\; tf''(t) \;,$$
where f'' denotes the second derivative of f. What is the matrix of φ with respect to the basis $\{1, t, t^2, t^3\}$ of V?

2. Let $\mathbf{R}^4$ denote 4-dimensional column space over the field $\mathbf{R}$ of real numbers. Let S be the subset of $\mathbf{R}^4$ consisting of the two vectors $v_1 = (2, -1, -1, 1)$ and $v_2 = (1, -2, 4, 2)$, and let W be the subspace of the dual space of $\mathbf{R}^4$ spanned by the two linear forms $f_1(x) = x_1 - 2x_2 + 3x_3 - x_4$ and $f_2(x) = 2x_1 - x_3 + x_4$.
   (a) Find a basis of the the pre-annihilator of W.
   (b) Find a basis of the annihilator of S.

3. Let F be a field, and let $P(t)$ be a member of the ring $F[t]$ of polynomials with coefficients in F. What is the dimension of the quotient space
$$F[t]/P(t)F[t] \;?$$

4. Let U be the set of matrices A in the vector space $\mathrm{M}_3(F)$ (of all 3×3 matrices in the field F) for which $\operatorname{trace}(A) = 0$.
   (a) Show that $\operatorname{trace} : \mathrm{M}_3(F) \longrightarrow F$ is a linear map.
   (b) What is the image of $\operatorname{trace} : \mathrm{M}_3(F) \longrightarrow F$?
   (c) Explain why U is a linear subspace of $\mathrm{M}_3(F)$.
   (d) Find the dimension of U without first finding a basis of U.
   (e) Find a basis of U.

5. For $x, y \in F^n$ let
$$B(x, y) \;=\; \sum_{i=1}^{n} x_i y_i \quad .$$
Observe that for each y the map $x \mapsto B(x, y)$ is a linear form on V, and, therefore, the map $y \mapsto B(\;, y)$ is a linear map $\lambda :\; V \longrightarrow V^*$.
   (a) Prove that this map λ is an isomorphism.
   (b) Does the construction of λ involve choice?

6. Let V be a finite-dimensional vector space over a field F, and let T be a subset of the dual space V^*. Recall that T has a pre-annihilator that is a subspace of V and also an annihilator that is a subspace of the second dual V^{**}. If α_V denotes the natural isomorphism $V \longrightarrow V^{**}$, prove that the image under α_V of the pre-annihilator of T is the annihilator of T.