Advanced Linear Algebra
Math 424/524

Assignment No. 4

Due December 2, 2002

1. Let V be a n-dimensional vector space over a field F, $\mathbf{v}$ be a basis of V, and $\mathbf{w}$ the dual basis of V^*. Let $\mathbf{v}'$ be another basis of V and $\mathbf{w}'$ the basis of V^* dual to it. If the change of basis between $\mathbf{v}$ and $\mathbf{v}'$, regarded as rows of vectors, is given by $\mathbf{v}' = \mathbf{v}A$ for an $n \times n$ matrix A and if also for V^* one has $\mathbf{w}' = \mathbf{w}B$, how is the matrix B related to the matrix A?

2. Let $f_j : V_j \longrightarrow W_j$ for $j = 1, 2$ be linear maps.
   (a) Explain briefly why there is a unique linear map
   $$f_1 \otimes f_2 : V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2$$
   for which
   $$(f_1 \otimes f_2)(v_1 \otimes v_2) \;=\; f_1(v_1) \otimes f_2(v_2) \quad \text{whenever} \quad v_1 \in V_1,\ v_2 \in V_2 \quad .$$
   (b) If for chosen bases in V_j, W_j, the linear map f_j has the matrix
   $$\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \text{ for } j \;=\; 1, 2 \ ,$$
   what is the matrix of $f_1 \otimes f_2$ with respect to suitable orderings of the product bases for $V_1 \otimes V_2$ and $W_1 \otimes W_2$?

3. Let $\mathbf{R}$ denote the field of real numbers. What more familiar description may be used to describe the d^{th} symmetric power $S^d((\mathbf{R}^2)^*)$ of the dual space of the real plane $\mathbf{R}^2$ to a student who has completed the calculus sequence?

4. Let $\mathbf{Q}$ denote the field of rational numbers, $P(t)$ the polynomial $t^5 + 5t^4 + 6t^3 + 7t^2 + 8t + 9$ in $\mathbf{Q}[t]$, and let
   $$V \;=\; \mathbf{Q}[t]/P(t)\mathbf{Q}[t] \quad .$$
   (a) Prove that V has dimension 5 over $\mathbf{Q}$ by exhibiting a basis of V.
   (b) If π denotes the quotient map from $\mathbf{Q}[t]$ to V, and τ denotes the linear endomorphism of $\mathbf{Q}[t]$ defined by $\tau(F(t)) = tF(t)$ for $F \in \mathbf{Q}[t]$, explain briefly why there is a linear endomorphism φ of V such that $\varphi \circ \pi = \pi \circ \tau$.
   (c) Find the matrix of φ with respect to the basis of V determined in response to the first part of this exercise.
   (d) What is the characteristic polynomial of the matrix found in the previous part of this exercise?

5. Let V and W be vector spaces over a field F. There is a natural linear map
   $$\rho : V \otimes W \longrightarrow \operatorname{Hom}(V^*, W)$$
   that is uniquely determined by specifying for $v \in V, w \in W$
   $$\rho(v \otimes w) \;=\; \{f \mapsto f(v)w \ \text{ for } \ f \in V^*\} \quad .$$
   Prove that ρ is an isomorphism if V and W are finite-dimensional.