Advanced Linear Algebra
Math 424/524

Assignment No. 3

Due November 13, 2002

1. Find a 2×2 real diagonal matrix D for which there exists a matrix U that is orthogonal relative to the standard inner product (the "dot" product) on $\mathbf{R}^2$ and satisfying
$$U^t \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} U \;=\; D \quad .$$

2. Find an invertible 2×2 matrix U of rational numbers and a *rational* diagonal matrix D such that
$$U^t \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} U \;=\; D \quad .$$

3. When $2 \neq 0$ in the field F, find an invertible 2×2 matrix U in F such that
$$U^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U \;=\; \begin{pmatrix} 1/9 & 0 \\ 0 & -1/9 \end{pmatrix} \quad .$$

4. Find a 3×3 rational matrix that is orthogonal for the standard inner product on $\mathbf{R}^3$ with the property that none of its entries has absolute value 1.

5. Find an invertible matrix U such that
$$U^t \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} U \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad .$$

6. What conditions in the definition of *inner product* are not satisfied by the bilinear form Θ on the space $\mathrm{M}_n(\mathbf{R})$ of $n \times n$ real matrices defined by
$$\Theta(M, N) \;=\; \mathrm{trace}\,(MN) \quad .$$

7. For a field F in which $2 \;=\; 0$ give an example of a bilinear form on F^3 that is skew-symmetric but not alternating.

8. Let b be the bilinear form on F^3 given by
$$b(x, y) \;=\; x^t \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} y \quad .$$
   (a) Explain very briefly why b is dualizing.
   (b) Find the left orthogonal complement of U, i.e.,
   $$\left\{ v \in F^3 \mid b(v, u) \;=\; 0 \text{ for each } \; u \in U \right\}$$
   in each of the three cases when U is a coordinate axis.
   (c) When $2 \neq 0$ in F, find a symmetric bilinear form s and an alternating bilinear form a on F^3 such that $b \;=\; s + a$.