Advanced Linear Algebra
Math 424/524

Assignment No. 2

Due October 16, 2002

1. Prove that if two matrices of the same size over a field are both in reduced row echelon form and are also row equivalent, i.e., each may be obtained from the other by a finite sequence of elementary row operations, then they must be equal.

2. Let $\mathbf{R}$ denote the field of real numbers.

   (a) What is the linear subspace of $\mathbf{R}^n$ spanned by the set of columns $x \in \mathbf{R}^n$ having the property that every coordinate of x is non-zero?

   (b) What is the linear subspace of the vector space $\mathrm{M}_n(\mathbf{R})$ of $n \times n$ real matrices that is spanned by the set of invertible $n \times n$ matrices?

3. Let F be a field, and let V denote the vector space $F[t]$ of polynomials in the variable t with coefficients in F. Let a be a given element of F, and let U_a be the linear subspace of V consisting of all polynomials divisible by the polynomial $t - a$. Does the isomophism class of the quotient space V/U_a depend on the choice of a?

   Hint: Consider the linear map $s_a : F[t] \to F$ defined by $s_a(f) = f(a)$.

4. If F is any field, let V be the vector space $\mathrm{M}_n(F)$ of $n \times n$ matrices over F. For given $A, B \in V$ define an F-linear endomorphism $\varphi_{A,B}$ of V by

$$\varphi_{A,B}(M) \;=\; AMB \quad .$$

   (a) For what pairs A, B is the endomorphism $\varphi_{A,B}$ equal to 0?

   (b) Is every endomorphism of V equal to $\varphi_{A,B}$ for some pair A, B?

5. Let X be a vector space over a field F, and let ψ be a linear map from X to X for which

$$\psi \circ \psi \;=\; \psi \quad .$$

   Let V be the subspace of X that is the image of ψ, let j be the inclusion of V in X, and let $q : X \to V$ be the "projection" of X on V that yields the canonical factorization $\psi = j \circ q$ of ψ through its image.

   Define a subspace U of X and a linear map $p : X \to U$ so that, with $i : U \to X$ the inclusion of U in X, one has the relations among i, j, p, and q characterizing an isomorphism of X with the Cartesian product $U \times V$, i.e.,

$$\begin{aligned} pi &= 1 \\ pj &= 0 \\ qi &= 0 \\ qj &= 1 \\ ip + jq &= 1 \end{aligned}$$