Advanced Linear Algebra
Math 424/524

Assignment No. 1

Due October 2, 2002

1. Let F be any field, and let V be the vector space of all polynomials of degree at most d in the variable t with coefficients in F. Exhibit an explicit isomorphism between V and the column space F^n for suitably chosen n.

2. Let F be a field, and let F_n^m be the vector space of all $m \times n$ matrices with entries in F. If V and W are vector spaces over F, then $\mathrm{Hom}(V, W)$ denotes the set of all F-linear maps from V to W. $\mathrm{Hom}(V, W)$ is itself a vector space over F under pointwise addition of linear maps and pointwise multiplication of a linear map by a scalar from F.

   Define
   $$\Phi : \; F_n^m \longrightarrow \mathrm{Hom}(F^n, F^m)$$
   by defining $\Phi(M)$ to be the linear map for which
   $$(\Phi(M))(X) \;=\; MX$$
   for each $X \in F^n$. Show that Φ is linear. (Proof of the linearity of $\Phi(M)$ for given M is not asked here.)

3. Let $f : V \longrightarrow W$ be an injective linear map of vector spaces over the field F. Prove that if elements $v_1, v_2, \ldots, v_r$ in V are linearly independent, then $f(v_1), f(v_2), \ldots, f(v_r)$ are linearly independent elements of W.

4. Let $F[t]$ be the vector space of polynomials in one variable t over the field F, and let $D : F[t] \to F[t]$ be the map defined by [1]
   $$D\left(\sum c_j t^j\right) \;=\; \sum j c_j t^{j-1} \quad .$$
   (a) Show that $D(f \cdot g) \;=\; f \cdot D(g) + D(f) \cdot g$.
   (b) Compute the kernel and image of D when F is the real field $\mathbf{R}$.
   (c) Compute the kernel and image of D when F is the field $\mathbf{F}_2$ of integers mod 2.

5. How many *rational* scalars c are there for which the matrices
   $$\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix}$$
   are similar? Justify your answer.

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[1] In this expression j as an index is an integer. How does one interpret jc_j given that c_j is in F? As long as j is a non-negative integer, the meaning of jc_j is "c_j added to itself j times in the field". If $j < 0$, then jc_j is understood as the negative of $(-j)c_j$. Consistent with that j itself can be interpreted in F as $j \cdot 1$ where 1 denotes the multiplicative identity of F. Thus, in $\mathbf{F}_2$: $j = 0$ if j is even, while $j = 1$ if j is odd.