Transformation Geometry — Math 331

April 16, 2004

The Classification of Isometries of R^3: IV

Proposition. A rotation commutes with any translation parallel to its axis and also with the mirror reflection in any plane normal to its axis. A mirror reflection commutes with any translation parallel to its fixed plane.

Proof. This becomes a simple matrix calculation if one chooses coordinates strategically in each of the three cases.

Definition. The isometry of $\mathbf{R}^3$ that results from composing a mirror reflection with a non-identity translation parallel to the fixed plane of the mirror reflection is called a **mirror glide**.

Theorem. An orientation-reversing isometry of $\mathbf{R}^3$ having a fixed point is either a mirror reflection or a reflective rotation.

Proof. Choose Cartesian coordinates for $\mathbf{R}^3$ such that the origin of coordinates is one of the fixed points. In that coordinate system the isometry has the form $f(x) = Ux$ where U is a 3×3 orthogonal matrix with determinant -1. Since the product of the three eigenvalues of U must be -1, while every eigenvalue must have complex absolute value 1 with the product of two that are either equal or complex conjugate having product 1, -1 must be an eigenvalue. If the other two are both 1, then f is the mirror reflection in the plane that is normal to the eigenspace for the eigenvalue -1, i.e., the 2-dimensional eigenspace for the eigenvalue 1. If the other two are both -1, then f is a point reflection in the origin, which is a special case of a reflective rotation. In the remaining case the other two eigenvalues are non-real complex conjugates of complex absolute value 1 and, therefore, have the form $\cos\theta \pm i \sin\theta$ for some θ, and f is the reflective rotation obtained by composing rotation through the angle θ about the line that is the 1-dimensional eigenspace for the eigenvalue -1 with reflection in the plane through the origin normal to that axis.

Theorem. An orientation-reversing isometry of $\mathbf{R}^3$ having no fixed point is a mirror glide.

Proof. Let $f = Ux + v$ where U is orthogonal with $\det U = -1$. Since f has no fixed point, the fixed point equation $(1 - U)x = v$ has no solution, i.e., v is not in the column space of the matrix $1 - U$. Therefore, $1 - U$ has rank less than 3, and 1 is an eigenvalue of U. Since -1 is also an eigenvalue of U, 1 must be an eigenvalue of multiplicity 2, , and therefore $\sigma(x) = Ux$ is a mirror reflection and $f = \tau\sigma$ where τ is translation by v. Write $\tau = \tau''\tau'$ where tau' is a translation perpendicular to the fixed plane of σ and tau'' is a translation parallel to the fixed plane. Then $\sigma' = \tau'\sigma$ is a mirror reflection whose fixed plane is parallel to the fixed plane of σ, and, therefore, $f = \tau''\sigma'$ is a mirror glide.

Corollary. If an orientation-reversing isometry has at least one fixed point, then its set of fixed points is either a plane or a single point.

Assignment for Monday, April 19

1. Find a 3×3 orthogonal matrix U for which the corresponding linear isometry of $\mathbf{R}^3$ is the mirror reflection in the plane $x + y + z = 0$.

2. Show that if two of the edges of a tetrahedron[1] in $\mathbf{R}^3$ do not share any vertex of the tetrahedron, then those edges are segments of skew (non-parallel and non-intersecting) lines.

3. What type of isometry of $\mathbf{R}^3$ results from composing the rotations about two skew lines in the case where the sum of the two angles of rotation about the respective axes is not an integral multiple of 2π?

4. Find a $3{\times}3$ matrix U and a vector v in $\mathbf{R}^3$ such that the isometry f defined by $f(x) = Ux+v$ is the composition $\sigma_D\sigma_C\sigma_B\sigma_A$ of the four reflections in the faces of the tetrahedron $ABCD$ where the notation σ_X denotes the reflection in the face formed with the three vertices other than X when

$$A = (1,0,0)\,,\ B = (0,1,0)\,,\ C = (0,0,1)\,, \text{ and } D = (0,0,0) \quad .$$

[1] A *tetrahedron* is the set of barycentric combinations with non-negative coefficients of 4 barycentrically independent points.