Transformation Geometry — Math 331

April 14, 2004

The Classification of Isometries of R^3: III

Proposition. If a rotation of $\mathbf{R}^3$ is composed with the translation by a vector that is perpendicular to the axis of the rotation, the resulting isometry is a rotation with axis parallel to the original axis.

Proof. Since both rotations and translations are orientation-preserving, the composition is orientation-preserving, and the question of why it must be a rotation is that of why it must have a fixed point. Let Π be a plane that is perpendicular to the axis of the given rotation. The given rotation induces a rotation in Π, and, since the given vector may be viewed as a vector in Π, the effect of the composed isometry of $\mathbf{R}^3$ in the plane Π is an isometry of Π that is the composition of a rotation in Π and a translation in Π. Since in Π the composition of a rotation and a translation is a rotation, the point in Π that is the center of the latter rotation of Π is a fixed point of the composed isometry of $\mathbf{R}^3$, and, therefore, the composed isometry is a rotation. It is clear that the distance of this fixed point, i.e., the point of the axis of the new rotation in Π, from the axis of the given rotation is independent of which plane Π normal to the original axis is chosen and, therefore, that the axis of the new rotation is parallel to that of the original.

Definition. The isometry that results from composing a non-identity rotation with a non-identity translation parallel to its axis is called a **screw**.

Proposition. If a non-identity rotation is composed with the translation by a vector that is not perpendicular to the axis of the rotation, the resulting isometry is a screw.

Proof. Since the vector may be written as the sum of a (possibly zero) vector perpendicular to the axis with a non-zero vector parallel to the axis of the rotation, the translation in question can be written as the composition of the translations by these two other vectors. Composition of the rotation with the translation by the vector perpendicular to the axis, according to the previous propostion, yields a rotation with parallel axis. Then composing that rotation with the non-zero vector parallel to its axis yields a screw.

Proposition. An orientation-preserving isometry of $\mathbf{R}^3$ is either a translation, a rotation, or a screw.

Proof. If an orientation-preserving isometry has a fixed point, then by definition the isometry is a rotation and its set of fixed points is a line. If there is no fixed point, then with τ the translation moving 0 to $f(0)$ the isometry $f \circ \tau^{-1}$ has a fixed point and so is a rotation, possibly the identity. Therefore, f is a screw or a translation.

Corollary. A non-identity orientation-preserving isometry of $\mathbf{R}^3$ either has a line of fixed points or has no fixed point.

Definition. The isometry that results from composing a non-identity rotation with the mirror reflection in a plane perpendicular to the axis of the rotation is called a **reflective rotation**.

Assignment for Friday, April 16

1. Give specific geometric descriptions of the isometries $f(x) = Ux + v$ of $\mathbf{R}^3$ given by the following pairs of 3×3 orthogonal matrices U and vectors v.

   (a)
   $$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} .$$

   (b)
   $$U = \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix}, \quad v = \begin{pmatrix} 2/3 \\ 2/3 \\ 2/3 \end{pmatrix} .$$

   (c)
   $$U = \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix}, \quad v = \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix} .$$

2. What type of isometry of $\mathbf{R}^3$ results from composing the mirror reflections in (a) two intersecting planes? (b) two parallel planes?

3. Show that in the case where the angle of rotation is π a reflective rotation is a point reflection.

4. What type of isometry of $\mathbf{R}^3$ results from composing a rotation with the mirror reflection in a plane containing the axis of the rotation?