Transformation Geometry — Math 331

March 31, 2004

The Dual Projective Plane

A triple $(a, b, c) \neq (0, 0, 0)$ of real numbers can represent either a point in $\mathbf{P}^2$ or the coefficient vector in the equation for a line in $\mathbf{P}^2$. In both cases the geometric object, i.e., the point or the line, is unchanged if the triple is multiplied by a non-zero scalar.

Thus, the set $\mathcal{L}$ of all lines in $\mathbf{P}^2$ shares with $\mathbf{P}^2$ the property that its "points" are represented by homogeneous non-zero triples of real numbers. Another way of viewing this is to regard $\mathcal{L}$ as another copy of $\mathbf{P}^2$: the *dual* projective plane.

A key characteristic of this duality is, firstly, that a point in $\mathcal{L}$, i.e., a line in $\mathbf{P}^2$, is determined by the set of points in $\mathbf{P}^2$ that belong to it and, secondly, that a point in $\mathbf{P}^2$ is determined by the set of lines in $\mathbf{P}^2$ to which it belongs.

Moreover, a line in $\mathcal{L}$ is the set of "points" (a, b, c) in $\mathcal{L}$ — with each such "point" corresponding to the line in $\mathbf{P}^2$ having the equation $ax+by+cz = 0$ — satisfying a homogeneous linear equation $Aa + Bb + Cc = 0$ with coefficient vector $(A, B, C) \neq (0, 0, 0)$. But when (A, B, C) is viewed as the triple of homogeneous coordinates for a point of $\mathbf{P}^2$, one sees that this point of $\mathbf{P}^2$ lies on every line $ax + by + cz = 0$ in $\mathbf{P}^2$ for which (a, b, c) is a "point" on the line $Aa + Bb + Cc = 0$ in $\mathcal{L}$, and, moreover, since a point in $\mathbf{P}^2$ is determined by the set of lines in $\mathbf{P}^2$ on which it lies, the lines in $\mathbf{P}^2$ corresponding to "points" of the line in $\mathcal{L}$ with equation $Aa + Bb + Cc = 0$ are precisely the lines in $\mathbf{P}^2$ containing (A, B, C) regarded as a point of $\mathbf{P}^2$.

Definition. A pencil of lines in $\mathbf{P}^2$ is the set of all lines in $\mathbf{P}^2$ containing a given point of $\mathbf{P}^2$.

The preceding discussion makes it clear that a pencil of lines in $\mathbf{P}^2$ is essentially the same thing as a "line" in the dual projective plane. Another way to state this is to say that the projective plane dual to $\mathcal{L}$ (the dual of the dual), which is the set of lines in $\mathcal{L}$, is essentially the same thing as $\mathbf{P}^2$.

Exercises due Friday, April 2

1. Show that the set of lines in $\mathbf{P}^2$ stabilized under the translation of $\mathbf{R}^2$ by the vector $(3, 4)$, treated as a projective transformation when $x+y+z = 0$ is taken as the line at infinity, is a pencil of lines. What point of $\mathbf{P}^2$ is the point where the lines of the pencil are coincident?

2. If (a_1, a_2, a_3) and (b_1, b_2, b_3) are triples of homogeneous coordinates for two different points of $\mathbf{P}^2$, what is the significance for those points of the "point" of the dual projective plane with homogeneous coordinate vector
$$(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \quad .$$

3. Find all fixed points and stabilized lines of the projective transformation given by the matrix
$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \quad .$$

4. Find all fixed points and stabilized lines of the projective transformation given by the matrix
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \quad .$$
Indicate what pencils occur among these lines.

5. Explain why every point of $\mathbf{P}^2$ lies on at least one of the lines in every pencil of lines in $\mathbf{P}^2$.

6. Given a pencil of lines in $\mathbf{P}^2$ how many points of $\mathbf{P}^2$ lie on more than one of the lines in the pencil?