Transformation Geometry — Math 331

March 15, 2004

Where Parallel Lines Meet: The Line at Infinity

Since any two triples of homogeneous coordinates (x, y, z) and (x', y', z'), relative to an affine basis of $\mathbf{R}^2$, for a point of $\mathbf{R}^2$ must be non-zero scalar multiples of each other, i.e., must both be points other than the origin on the same line through the origin, it is clear that the *set* of all homogeneous triples (x, y, z) for a given point of $\mathbf{R}^2$ is a line through the origin in $\mathbf{R}^3$ except for its origin.

Because a triple (x, y, z) of homogeneous coordinates for a point in $\mathbf{R}^2$ must "scale" to a triple of barycentric coordinates for that point, the homogeneous coordinates must satsify the condition $x+y+z \neq 0$. That is, the point (x, y, z) of $\mathbf{R}^3$, aside from not being the origin of $\mathbf{R}^3$, must not lie in the plane $x + y + z = 0$. Because the latter equation has no constant term, the issue of whether a point $(x, y, z) \neq (0, 0, 0)$ lies in the plane $x + y + z = 0$ is the same for all of the points, other than the origin, on its line through the origin.

On the other hand, a point other than the origin in $\mathbf{R}^3$ like, for example, the point $(1, -2, 1)$ that lies in the plane $x + y + z = 0$ certainly determine a line through the origin, and, even though it cannot be a triple of homogeneous coordinates for a point of $\mathbf{R}^2$, one may ask what role it might have for the geometry of $\mathbf{R}^2$. For example, there are many planes through the origin in $\mathbf{R}^3$ that contain this point and its scalar multiples but also contain the homogeneous coordinates of points in $\mathbf{R}^2$. The plane $2x+y = 0$ and the plane $x-z = 0$ are two examples of these. The first of these equations is the homogeneous equation for the line in $\mathbf{R}^2$ with Cartesian equation $2x+y = 0$, and the second is the homogeneous equation for the parallel line with Cartesian equation $2x + y = 1$. While, the lines in $\mathbf{R}^2$ with Cartesian equations $2x + y = 0$ and $2x + y = 1$ do not meet in $\mathbf{R}^2$, there is a sense in which one may regard the triple $(1, -2, 1)$ as homogeneous coordinates of a "point at infinity" where the lines do meet.

Definition. The *projective plane* is the set $\mathbf{P}^2$ whose "points" are the lines through $(0, 0, 0)$ in $\mathbf{R}^3$.

Thus, a point in $\mathbf{P}^2$ may be represented by *any* non-zero (x, y, z) triple of coordinates for a point on the line through the origin of $\mathbf{R}^3$ that *is* the corresponding element of $\mathbf{P}^2$ whether or not it satisfies the condition $x + y + z \neq 0$.

Definition. A *line* in $\mathbf{P}^2$ is the set of elements in $\mathbf{P}^2$ lying in a plane through the origin in $\mathbf{R}^3$. A line in $\mathbf{P}^2$ consists of the points represented by homogeneous coordinates (x, y, z) satisfying the homogeneous linear equation $ax + by + cz = 0$, given by a triple $(a, b, c) \neq (0, 0, 0)$, that is also the equation of the corresponding plane in $\mathbf{R}^3$. Of course, a line in $\mathbf{P}^2$ depends on its triple of coefficients only up to multiplication by a non-zero scalar.

Proposition 1. A line $ax + by + cz = 0$ in $\mathbf{P}^2$ is affine, i.e., is the homogeneous form of a line in $\mathbf{R}^2$ if and only if a, b, c are not all equal. (See the assignment due March 12.)

Proposition 2. There is one and only one line through two different points of $\mathbf{P}^2$.

Proposition 3. Any two different lines in $\mathbf{P}^2$ meet in one and only one point of $\mathbf{P}^2$.

Corollary. Every line in the affine plane contains one and only one point on the line at infinity (the line $x+y+z = 0$ in $\mathbf{P}^2$), and two lines in the affine plane share the same point on the line at infinity if and only if they are parallel lines in the affine plane.

Exercises due Wednesday, March 17

1. Find the point in the projective plane where the line $4x + 3y + 6z = 0$ meets the line $6x + 11y + 9z = 0$.

2. Find a homogeneous equation for the line in $\mathbf{P}^2$ through the two points $(2, -3, 1)$ and $(1, 2, 3)$. Give a Cartesian equation for this line as an affine line.

3. Find the point in the projective plane $\mathbf{P}^2$ where the line $2x + 3y = 6$ in the affine plane meets the parallel line $2x + 3y = 0$.

4. Prove Propositions 2 and 3 above. Is there a common theme in your two arguments?

5. In another course you learned that the projective plane $\mathbf{P}^2$ was the set of lines in $\mathbf{R}^3$ through the origin but that a point in the affine plane represented as (x, y) with Cartesian coordinates is included in $\mathbf{P}^2$ as the line in $\mathbf{R}^3$ through the origin and the point $(x, y, 1)$. Does this mean that there are two different versions of the projective plane?