Transformation Geometry — Math 331

March 3, 2004

Order 2 transformations

Definition. An affine transformation f of $\mathbf{R}^n$ has *order* k if $k > 0$, the identity results from composing it with itself k times, and k is the smallest positive integer with that property.

When f has order k, clearly f is inverted by f^{k-1}. In particular, an affine transformation of order 2 is a transformation not the identity that is its own inverse.

Proposition 1. *An affine transformation $f(x) = Ax + b$ is of order 2 if and only if $A^2 = 1$ and $Ab = -b$.*

Proposition 2. *If $f(x) = Ax + b$ is of order 2, then A cannot be the identity matrix.*

Proposition 3. *If $f(x) = Ax + b$ is of order 2, then* $\det(A) = \pm 1$.

Proposition 4. *When $n = 2$ if f is orientation-preserving and of order 2, then f must be the half turn about some point.*

Proposition 5. *When $n = 2$ if $f(x) = Ax + b$ is orientation-reversing and of order 2, then*

a. The characteristic polynomial of A must be $t^2 - 1$.

b. f must have a line of fixed points.

Comment on past exercises

February 27, No. 1 Represent the two rotations as compositions of reflections in lines through their centers where the line between the two centers is used twice to obtain the required rotation as the composition of reflection in the line through $(1,0)$ with elevation $5\pi/8$ followed by reflection in the line through $(0,1)$ with elevation $-\pi/6$. The center of the required rotation is the point where these two lines meet, which is

$$\left(\frac{3-\sqrt{3}\left(\sqrt{2}-1\right)}{4-\sqrt{2}\left(\sqrt{3}-1\right)}, \frac{\sqrt{3}\left(\sqrt{3}-1\right)+\sqrt{2}\left(2-\sqrt{3}\right)}{4-\sqrt{2}\left(\sqrt{3}-1\right)} \right) .$$

March 1, No. 1 The line segment from X to $f(X)$ must be perpendicular to the given line containing A and B, and the midpoint of that segment must lie on the given line. Therefore $\frac{X+f(X)}{2} = sA + tB$ with $s = 1-t$. Of course, t depends on X, and may be determined by using the condition of perpendicularity in the form $(X - A - t\,(B-A)) \cdot (B-A) = 0$. One finds

$$f(X) = 2\left\{ \frac{(B-X)\cdot(B-A)}{(B-A)\cdot(B-A)} \right\} A + 2\left\{ \frac{(X-A)\cdot(B-A)}{(B-A)\cdot(B-A)} \right\} B - X .$$

Exercises due Friday, March 5

1. Let f be the affine transformation defined by

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} .$$

   (a) Show that f is orientation-reversing and of order 2.

   (b) Show that for any point x in $\mathbf{R}^2$ the vector from x to $f(x)$ is parallel to the vector $(2,1)$.

   (c) Find the line of fixed points of f.

2. Write proofs for as many as you can of the propostions above.