Transformation Geometry — Math 331

February 27, 2004

Desargues' Theorem

Theorem. Let A, A', B, B', C, C' are six points in a plane for which both A, B, C and A', B', C' are non-collinear triples and let BC meet $B'C'$ at D, CA meet $C'A'$ at E, and AB meet $A'B'$ at F. Then the three lines AA', BB', and CC' are coincident if and only if the three points D, E, and F are collinear.

Proof. Assume that P is a common point of the three lines. Since P lies on AA', it is a barycentric combination of A and A': $P = aA + a'A'$ with $a + a' = 1$. Likewise $P = bB + b'B'$ with $b + b' = 1$, and $P = cC + c'C'$ with $c + c' = 1$. From the relation $bB + b'B' = cC + c'C'$ follows $(b - c)D = bB - cC = -b'B' + c'C'$. Likewise also $(c - a)E = cC - aA = -c'C' + a'A'$, and $(a - b)F = aA - bB = -a'A' + b'B'$. Summing the last three relations yields

$$(b - c)D + (c - a)E + (a - b)F = 0 ,$$

which is a non-trivial weight 0 linear relation among the points D, E, and F. Therefore, D, E, F are collinear.

Proof of the converse will only be sketched. If D, E, and F are collinear, then let P be the point where BB' meets CC', Q the point where CC' meets AA', and R the point where AA' meets BB'. One wants to know that these three points are the same. If each of D, E, and F is represented as a barycentric combination of both pairs of points on the corresponding intersecting lines, then three relations among the original six points are observed. Those relations may be re-arranged to produce homogeneous combinations of both B, B' and C, C' representing P, of both C, C' and A, A' representing Q, and of both A, A' and B, B' representing R. Equalities among P, Q, R follow from parallelism in the two different homogeneous coefficient pairs involving A, A', B, B', and C, C' that results from making use of the fact that F is a barycentric combination of D and E, while weight 0 linear combinations of A, B, C and of A', B', C' must be trivial.

Exercises due Monday, March 1

1. Let A and B be different points in $\mathbf{R}^2$, l the line through them, and σ the reflection in l. For a given point X in $\mathbf{R}^2$ write a formula expressing the point $\sigma(X)$ as a barycentric combination of the points A, B, and X.

2. Given rotations about two different points in the plane for which the sum of the two angles of rotation is not an integer multiple of 2π, describe a procedure, based on diagramming isometries, for constructing the center of the composition of the two rotations.

3. Let A, B, and C be three non-collinear points in $\mathbf{R}^2$ and σ_a, σ_b, and σ_c, respectively, the reflections in the lines BC, CA, and AB, respectively. Show that if $\angle C \neq \pi/2$, and if F is the foot of the altitude from C to the side AB, then there is a point G on AB such that $\sigma_c \circ \sigma_b \circ \sigma_a$ is the composition $h \circ \sigma$ where h is the half turn about F and σ is reflection in the line CG.