Transformation Geometry — Math 331

February 23, 2004

Diagramming Isometries

Since a rotation may always be represented as the composition of reflections in two intersecting lines and a translation may always be represented as the composition of reflections in two parallel lines and since a glide reflection is the composition of a reflection and a translation and since every isometry is one of the above, it follows that every isometry other than the identity is the product (via composition of transformations) of one, two, or three reflections.

A diagram consisting of a finite number of lines in the plane may be regarded as representing the isometry that is the product of the reflections in those lines once it is stipulated in which order the reflections should be composed.

In particular

- A reflection may be represented by a single line. The line is its axis.
- A rotation may be represented by a pair of intersecting lines. The center of the rotation is the point of intersection, and the angle of rotation about the center is twice the angle from the first line, i.e., the line in which reflection is first applied to form the isometry, to the second line.
- A translation may be represented by a pair of parallel lines. The lines are both perpendicular to the direction of translation, and the vector of translation is twice the vector (of shortest length) drawn from a point on the first line to the second line.
- A glide reflection may be represented by a triple of lines of which two are parallel and the other is perpendicular to both of those.

Exercises due Wednesday, February 25

1. Let U be the 2×2 matrix
$$U = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} .$$
   (a) Explain in one sentence why the transformation f defined by $f(x) = Ux$ is a reflection.
   (b) Compute $f(1, -2)$.
   (c) What line is the axis of the reflection f?
   (d) What type of isometry is the transformation g defined by the formula
$$g(x) = Ux + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
   (e) What type of isometry is the transformation g defined by the formula
$$g(x) = Ux + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
2. Give line diagrams illustrating the following
   (a) The composition of a rotation followed by a translation is a rotation.
   (b) The composition of a translation followed by a rotation is a rotation.
   (c) The composition of a rotation followed by the reflection in a line through the center of the rotation is a reflection.
3. What rather special isometry is diagrammed by a pair of mutually perpendicular lines?
4. If a reflection is followed by a translation, is it necessarily true that the composite is a glide reflection?