Transformation Geometry — Math 331

February 20, 2004

Discussion

- **Theorem.** Every orientation-preserving isometry of $\mathbf{R}^2$ with a fixed point is a rotation.

  Proof. Let f be a given orientation-preserving isometry of $\mathbf{R}^2$ with fixed point c. Let τ be "translation by c", i.e., $\tau(x) = x + c$. Then the isometry $g = \tau^{-1} \circ f \circ \tau$ has the property $g(0) = 0$. Since g is an affine map that fixes the origin, g must be a linear transformation of $\mathbf{R}^2$ that is distance-preserving. Therefore, $g(x) = Ux$ for some 2×2 orthogonal matrix U. By an exercise in the previous assignment U must be one of the matrices formed using $\cos\theta$ and $\sin\theta$ for some value of θ, and since g is orientation-preserving, $\det U > 0$ with the result that U must be the specific matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad .$$

  Therefore, g is the rotation about the origin through the angle θ, and f is the rotation about the point c through the angle θ.

- **Theorem.** Every orientation-reversing isometry of $\mathbf{R}^2$ with a given fixed point is the reflection in some line containing the fixed point.

  Proof. The argument is very similar to the preceding argument except that the 2×2 orthogonal matrix U satisfies $\det(U) < 0$ since the isometry is orientation-reversing, and, therefore,

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} ,$$

  which is the matrix of reflection in the line through the origin forming the angle $\theta/2$ with the positive first coordinate axis.

- **Proposition** Every rotation of $\mathbf{R}^2$ is the composition of the reflections in two lines passing through its center.

  Proof. For example, let σ_1 be reflection in the horizontal line through the center and let σ_2 be reflection in the line through the center forming angle $\theta/2$ with the horizontal, where θ is the angle of rotation about the center. Then $\sigma_2 \circ \sigma_1$ is the given rotation.

Exercises due Monday, February 23

1. Prove: If an isometry f of the plane is a rotation about the point p, then for every point x in the plane p must lie on the perpendicular bisector of the line segment from x to $f(x)$.
2. Show that every translation of $\mathbf{R}^2$ is the composition of the reflections in two parallel lines that are perpendicular to the direction of translation.
3. Show that the composition of a rotation with the reflection in a line through the center of the rotation is another such reflection.
4. Let A, B, C, and P be four points in a plane, no three of which are collinear. Let PA meet BC at D, PB meet CA at E, and PC meet AB at F. Prove that D, E, and F are not collinear.