Transformation Geometry — Math 331

February 18, 2004

Discussion

- Granted the classification of isometries of the plane presented with the assignment due February 13, one sees that which of the four classes a plane isometry falls in is determined by the answers to two questions:
    1. whether or not the isometry has a fixed point.
    2. whether or not the isometry preserves orientation.
- **Definition.** A point x is a *fixed point* of a transformation f of $\mathbf{R}^n$ if $f(x) = x$.
- **Examples.** A (non-identity) rotation has a single fixed point — its center. The set of fixed points of a reflection is a line — its axis.
- **Definition.** An affine transformation $f(x) = Ux + v$ of $\mathbf{R}^n$ is *orientation-preserving* if $\det U > 0$ and *orientation-reversing* if $\det U < 0$.
- **Remark.** One might be inclined to seek a more elemental definition of orientation behavior, but such is not easy to find. The preceding definition, however, may certainly be generalized from affine to differentiable transformations by reference to the determinant of the "Jacobian matrix" of the transformation.

Exercises due Friday, February 20

1. Let f be the rotation of the plane about the point $(-1, 3)$ counterclockwise through the angle $2\pi/3$. Find a matrix U and a vector v such that such that $f(x) = Ux + v$ for all points x in the plane.

2. Show that a 2×2 orthogonal matrix U must, for some value of t, be one or the other of:
$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \quad .$$
   In each case describe the isometry of $\mathbf{R}^2$ given by the formula $f(x) = Ux$.

3. Let A, B, and C be points in $\mathbf{R}^3$ with
$$A = (3, -2, 4) \quad B = (2, -1, 0) \quad \text{and} \quad C = (1, -3, 2)$$
   and let P be the point where the angle bisectors of $\Delta\, ABC$ meet. Find the area of $\Delta\, ABP$.

4. Let r be a given real number and $a = (a_1, a_2)$. Define a transformation f of $\mathbf{R}^2$ by the formula
$$f(x) = a + r(x - a) = (1 - r)a + rx \quad .$$
   (a) Show that f is affine by finding a matrix U and a vector v such that $f(x) = Ux + v$.
   (b) What effect does f have on the distance? That is, how does the distance from $f(x)$ to $f(y)$ relate to the distance from x to y?
   (c) Does f have fixed points?
   (d) Is f orientation-preserving or orientation-reversing?