Transformation Geometry — Math 331

February 9, 2004

Discussion

- **Definition:** By *affine basis* of $\mathbf{R}^n$ is meant a sequence $P_0, P_1, \ldots, P_n$ of $n+1$ barycentrically independent points of $\mathbf{R}^n$.

- **Proposition.** Any point of $\mathbf{R}^n$ is uniquely representable as a barycentric combination of the points in a given affine basis of $\mathbf{R}^n$.

  Proof. Given P and an affine basis $P_0, P_1, \ldots, P_n$ use the fact from linear algebra that the vectors $P_1 - P_0, \ldots, P_n - P_0$ form a linear basis of $\mathbf{R}^n$ and that $P - P_0$ is uniquely a linear combination of those vectors.

- **Terminology.** The coefficients used to represent a point P as a barycentric combination of $P_0, P_1, \ldots, P_n$ are called *barycentric coordinates* or *affine coordinates* of P with respect to (or relative to) $P_0, P_1, \ldots, P_n$.

- **Definition.** Any sequence of $n+1$ numbers that is proportional to (a non-zero multiple of) a sequence of barycentric coordinates of P with respect to an affine basis $P_0, P_1, \ldots, P_n$ is a sequence of *homogeneous coordinates* of P with respect to (or relative to) $P_0, P_1, \ldots, P_n$.

  Example. (a, b, c) is a sequence of homogeneous coordinates for the point where the angle bisectors of $\Delta\, ABC$ meet relative to the vertices of the triangle since $1/(a+b+c)$ times that triple is the corresponding sequence of barycentric coordinates.

- **Theorem.** The point where the three altitudes of a triangle meet has homogeneous coordinates relative to the vertices of the triangle given by the areas of the three sub-triangles formed by that point and the three vertices when all of the angles in the triangle are acute.

Exercises due Wednesday, February 11

1. Let A, B, and C be the points

$$A \;=\; (0, -1)\;,\quad B \;=\; (3, 4)\;,\quad C \;=\; (-1, 1)\quad .$$

   (a) Find the point P where the three altitudes of $\Delta\, ABC$ meet.

   (b) Find the areas of the three triangles: $\Delta\, BCP$, $\Delta\, CAP$, and $\Delta\, ABP$.

   (c) Find a triple of homogeneous coordinates for P relative to A, B, and C.

2. Show that three distinct points A, B, and C are collinear if there is a triple of numbers (u, v, w), not all zero, of weight 0, i.e., $u + v + w = 0$, such that $uA + vB + wC = 0$.

3. Let $f(x) = Ax$ be the linear transformation of the plane where A is the matrix

$$A \;=\; \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\quad .$$

   (a) What points x of the plane are “fixed” by f, i.e., satisfy $f(x) = x$?

   (b) What lines in the plane are carried by f to other lines?

   (c) What lines L in the plane are “stabilized” by f, i.e., satisfy the condition that $f(x)$ is on L if x is on L?

4. Find homogeneous coordinates relative to the vertices of a given triangle for the point where the three perpendicular bisectors of the sides of the triangle meet.

   Hint: Use the fact that the perpendicular bisectors are the altitudes of the triangle whose vertices are their feet (i.e., the midpoints of the three sides).