Transformation Geometry — Math 331

January 30, 2004

Discussion

Theorem. Let A, B, and C be non-collinear points, and let D, E, and F, respectively, be given points on the lines BC, CA, and AB, respectively, with corresponding barycentric representations $D = d'B + d''C$, $E = e'C + e''A$, and $F = f'A + f''B$ and none of D, E, or F equal to any of A, B, or C. Then the three lines AD, BE, and CF meet in a common point if and only if

$$\left(\frac{d'}{d''}\right)\left(\frac{e'}{e''}\right)\left(\frac{f'}{f''}\right) = 1 \ . \tag{1}$$

Proof. Note that none of the numbers $d', d'', e', e'', f', f''$ can be zero under the given hypotheses. If $P = uA + vB + wC$ (with $u + v + w = 1$) is a common point on AD, BE, and CF, then by the principle of preservation of proportionality in barycentric combinations (Problem 2, Assignment due January 28) one has

$$d' = \frac{v}{v+w}, \quad d'' = \frac{w}{v+w}, \quad e' = \frac{w}{w+u}, \quad e'' = \frac{u}{w+u}, \quad f' = \frac{u}{u+v}, \quad \text{and} \quad f'' = \frac{v}{u+v} \quad .$$

Therefore, $d'/d'' = v/w$, $e'/e'' = w/u$, and $f'/f'' = u/v$, and then clearly the product of these three values is 1.

Conversely, assume that $(d'/d'')(e'/e'')(f'/f'') = 1$. The question now is whether there is a triple of numbers (u', v', w'), none zero, such that the three pairs (v', w'), (w', u'), and (u', v'), respectively, are parallel to (d', d''), (e', e''), and (f', f''). For if that is the case, then with

$$u = \frac{u'}{u' + v' + w'}, \quad v = \frac{v'}{u' + v' + w'}, \quad \text{and} \quad w = \frac{w'}{u' + v' + w'}$$

the lines drawn from the point $P = uA + vB + cW$ to the vertices meet the sides of the triangle in the points D, E, F by the principle of preservation of proportionality in barycentric combinations. For the existence of such u', v', w' let

$$u' = f', \quad v' = f'', \quad \text{and} \quad w' = \frac{d''f''}{d'} \quad .$$

Then clearly (u', v') is parallel to (f', f'') and $(v', w') = (f''/d')(d', d'')$ is parallel to (d', d''). Finally,

$$(w', u') = \left(\frac{d''f''}{d'}, f'\right) = \frac{f'}{e''}(e', e'') \ ,$$

by formula (1) and, therefore, (w', u') is parallel to (e', e'').

Exercises due Monday, February 2

1. For what values of c are the three points $(c, -1)$, $(3, 2)$, and $(-2, 1)$ barycentrically dependent? What is the geometric significance of this issue?

2. Prove **Ceva's Theorem:** If P is a point in the interior of the triangle determined by three non-collinear points A, B, and C and if D, E, and F, respectively, are the points where the lines from P to the points A, B, and C, respectively, meet the lines BC, CA, and AB, respectively, then one has the relation

$$\frac{|BD|}{|DC|}\frac{|CE|}{|EA|}\frac{|AF|}{|FB|} = 1$$

   among the lengths of the six line segments.

3. Prove: If A, B, and C are three non-collinear points in a plane and l is a line in that plane meeting the lines BC, CA, and AB, respectively in points D, E, and F, respectively, having barycentric coordinate pairs (d', d''), (e', e''), and (f', f''), respectively, with respect to A, B, C, then one has the relation

$$\left(\frac{d'}{d''}\right)\left(\frac{e'}{e''}\right)\left(\frac{f'}{f''}\right) = -1 \quad .$$