Abstract Algebra (Math 327)
Assignments

Spring Semester, 2003

Assignments are listed by the **date due**.

PDF and DVI (requires *TeX*[1] software) versions of this page are available for printing.

Most of these assignments are simply exercises designed to prepare you for the quizzes and the written assignments. Those which are to be submitted as written assignments are so labeled. While you may find it helpful to discuss the exercises with others, no collaboration is permitted on the written assignments.

Fri., May. 9:

Final Examination, 10:30 – 12:30

Question. Will the final be based on the whole course or just on the material from after the midterm?
Answer. The whole course with emphasis on material covered since the midterm.

Thu., May. 8:

Office Hours: 3:00 – 4:00

Wed., May. 7:

Office Hours: 4:00 – 5:00

Mon., May. 5:

Last Regular Meeting of the Course: Bring review questions.
Written Assignment No. 5 is due:

1. List 5 groups that represent different isomorphism classes among the groups of order 8.
2. Determine the number of different ring isomorphism classes that arise among the four rings given as
$$(\mathbf{Z}/2\mathbf{Z})\,[t]/f(t)\,(\mathbf{Z}/2\mathbf{Z})\,[t]$$
for each of the four polynomials $f(t)$ of degree 2 over the field $\mathbf{Z}/2\mathbf{Z}$
3. Decompose the polynomial $x^{12} - 1$ into irreducible factors over the field $\mathbf{Z}/5\mathbf{Z}$.
4. Explain why every group of order 175 must contain a non-trivial proper normal subgroup. Can a group of order 175 contain a non-normal subgroup?
5. Write a proof of the following proposition:

   Proposition. *A finite abelian group of order n is cyclic if and only if for each integer $k > 0$ dividing n there is one and only one order k subgroup of G.*

   In your argument you may feel free to cite the theorem stating that every finite abelian group is isomorphic to the direct product of cyclic groups whose orders successively divide each other.

Fri., May. 2:

(This is an exercise, not an assignment to be submitted.)

The group $G = \mathrm{O}_3(\mathbf{Z})$ of 3×3 orthogonal integer matrices, i.e., integer matrices inverted by their transposes, is a group of order 48.

1. Verify that G has order 48.
2. What isomorphism class is shared by the Sylow 3-subgroups of G?
3. Identify a normal subgroup of index 2 in G.
4. Find a commutative normal subgroup of index 6 in G.

[1]URI: http://www.tug.org/

5. What is the largest order of any element of G?
6. Determine the number of Sylow 3-subgroups of G.
7. Determine the number of Sylow 2-subgroups of G.
8. Determine the isomorphism class that is shared by the Sylow 2-subgroups of G.

Wed., Apr. 30:

Read: § 4.4

Exercises:

392: 33, 34
222: 11, 17
229: 4, 6

Mon., Apr. 28:

Read: § 4.3

Exercises:

381: 18
392: 30, 31
216: 12
222: 6, 10, 12, 13

Fri., Apr. 25:

Read: § 4.2

Exercises:

334: 40
373: 20
381: 14, 15
391: 17, 25
216: 7
222: 1 – 3, 5

Wed., Apr. 23:

Written Assignment No. 4 is due:

1. Find all units in the ring $(\mathbf{Z}/7\mathbf{Z})\,[x]$.
2. Decompose the polynomial $x^8 - 1$ into irreducible factors in the ring $(\mathbf{Z}/2\mathbf{Z})\,[x]$.
3. Let A denote the ring $\mathbf{Z} + \mathbf{Z}\sqrt{-1}$ of Gaussian integers. Recall that A is a Euclidean domain, and, therefore, a principal ideal domain. Find an element $\alpha \in A$ for which
$$\alpha A \;=\; \mathbf{Z}2 + \mathbf{Z}(5 - \sqrt{-1}) \quad .$$
4. Let $m \geq 0$ be an integer, and let R denote the ring $\mathbf{Z} + \mathbf{Z}\sqrt{-5}$. Let T_m denote the additive subgroup of R given by
$$T_m \;=\; \mathbf{Z}7 + \mathbf{Z}(m - \sqrt{-5}) \quad .$$
   (a) Find the smallest value of $m \geq 0$ for which T_m is an ideal in R.
   (b) Find the isomorphism class of the quotient ring R/T_m for the value of m obtained in the previous part.

Wed., Apr. 16:

This reading assignment returns to the theory of groups:

Read: § 4.1

Exercises:

334: 37, 38
373: 13, 15, 16
380: 8 – 10
390: 4, 6, 13
215: 1, 3

Mon., Apr. 14:

Read: § 8.1

Exercises:

333: 31, 32, 34
342: 24, 26 – 28
366: 28, 30, 31
373: 9, 12
380: 5, 7
390: 1 – 3

Fri., Apr. 11:

Read: § 7.3

Exercises:

307: 30 – 32
333: 27 – 30
342: 18, 19, 21 – 23
366: 18 – 21, 25, 26
373: 7
380: 1 – 3

Wed., Apr. 9:

Read: § 7.2

Exercises:

296: 28, 29
307: 23, 24, 27 – 29
333: 20, 21, 23, 26
342: 15 – 17
366: 12, 14
373: 1 – 3

Mon., Apr. 7:

Read: § 7.1

Exercises:

296: 24, 25, 27
307: 16, 25, 26
332: 15, 17, 18
341: 7, 8, 14
366: 1 – 6, 9, 10

Note: Due to icy conditions on April 4, the date for re-submission of problem 2 from Written Assignment No. 3 is postponed until Monday, April 7.

Also: A unannounced quiz that had been scheduled for April 4 has been postponed until Monday, April 7.

Fri., Apr. 4:

Read: § 6.2

Exercises:

295: 12 – 14, 16, 20, 22, 23
306: 2, 7, 9, 14
331: 5 – 9, 12 – 14
341: 2, 4, 5

Wed., Apr. 2:

Read: § 5.6

Written Assignment No. 3 is due:

1. Find a familiar group that is isomorphic to the quotient group of the group D_4 (the group of symmetries of the square) by its center.

2. Find a simple real-valued function $f(x, y)$ of two variables for which the map $\phi : \mathcal{H}_3(\mathbf{R}) \to \mathcal{H}_3(\mathbf{R})$ defined by

$$\phi(x, y, t) \;=\; (uy, x, t + f(x, y))$$

is an automorphism of $\mathcal{H}_3(\mathbf{R})$ when $u = -1$, but show that for every function $f(x, y)$ this formula does not yield an automorphism of $\mathcal{H}_3(\mathbf{R})$ when $u = 1$.

Mon., Mar. 31:

Read: §§ 5.5, 6.1

Exercises:

284: 12, 14, 15
295: 1 – 11
331: 1 – 3

Fri., Mar. 28:

Read: § 5.4

Exercises:

269: 24
276: 19, 20, 24, 27
283: 1, 2, 4

Wed., Mar. 26:

Read: § 5.3

Exercises:

262: 32, 40, 42, 46 – 48, 50, 52
269: 18, 21, 23, 28
276: 2, 3, 5, 6 – 9, 12, 14, 15
and the following:
Without consulting the text write a proof of:
Proposition. *If G is a group, X a G-set, g an element of G, and x an element of X, then there is a relation of conjugacy between the isotropy group of the action of G on X at x and the isotropy group of the action at $g \cdot x$:*

$$G_{g \cdot x} \;=\; g\,(G_x)\,g^{-1} \quad .$$

Mon., Mar. 24:

Read: § 5.2

Exercises:

261: 21 – 28, 33, 35, 37, 38
268: 1 – 8, 11, 15 – 17
and the following:
Let $G = \mathrm{SL}_2(\mathbf{R})$ be the group of all 2×2 real matrices

$$M \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for which $\det(M) = 1$, i.e., $ad - bc = 1$. Let $\mathbf{H}$ be the upper-half plane, i.e., the set of all complex numbers $z = x + iy$ with $y > 0$ where $i = \sqrt{-1}$. $\mathbf{H}$ is a G-set under the action

$$M \cdot z \;=\; \frac{az + b}{cz + d} \quad .$$

1. Find the subgroup of G that acts trivially on $\mathbf{H}$, i.e., the set of all $g \in G$ such that $g \cdot z = z$ for all $z \in \mathbf{H}$.
2. Find the isotropy group G_i at i, i.e., the subgroup of all $g \in G$ such that $g \cdot i = i$.
3. Find the orbit in $\mathbf{H}$ of i, i.e., the set of all points in $\mathbf{H}$ of the form $g \cdot i$ for at least one $g \in G$. *Hint:* Consider the two special cases

$$(1)\;\; g \;=\; \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \;\text{(when } r > 0\text{)} \;\;\text{and}\;\; g \;=\; \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad .$$

Fri., Mar. 21:

Read: § 5.1

Exercises:

260: 1 – 20

and the following concerning the 3-dimensional Heisenberg group $\mathcal{H}_3(\mathbf{R})$, which is given by a "twisted" group law on $\mathbf{R}^3$:

$$(x_1, y_1, t_1) * (x_2, y_2, t_2) \;=\; (x_1 + x_2, y_1 + y_2, t_1 + t_2 + x_1 y_2) \quad .$$

1. Find the *center* of $\mathcal{H}_3(\mathbf{R})$.
2. Find the commutator $\xi\eta\xi^{-1}\eta^{-1}$ when $\xi = (x_1, y_1, t_1)$ and $\eta = (x_2, y_2, t_2)$.

Wed., Mar. 19:

Exercises:

138: 39, 40, 42, 46, 49
189: 39, 40
204: 18 – 20
208: 1

Mon., Mar. 17:

Midterm Test

Fri., Mar. 14:

Review Session: Bring questions.

Wed., Mar. 12:

Read: §§ 3.5, 3.6

Exercises:

135: 26 – 28, 30, 32, 33, 35, 44
188: 5 – 9, 17 – 21, 24, 25, 27, 32, 34, 35
202: 1 – 3, 8, 11 – 13, 15

Mon., Mar. 10:

Written Assignment No. 2

For each of the following propositions either provide a proof that it is true or provide evidence demonstrating that it is false.

1. Let $\phi : G \to G'$ be a group homomorphism. Show that if $|G|$ is finite, then $|\phi(G)|$ is finite and divides $|G|$.
2. Show that any group homomorphism $\phi : G \to G'$ where $|G|$ is prime must either be the trivial homomorphism or else be a one-to-one map.

Sat., Mar. 1 – Sun., Mar. 9

Recess

Fri., Feb. 28:

Finish reading § 2.4

Exercises:

177: 36 – 40
135: 9 – 11, 13, 14, 21, 22, 24
188: 1 – 4, 14

Wed., Feb. 26:

Read: § 3.3

Exercises:

169: 47, 49, 50, 52
177: 31 – 35
188: 13

Mon., Feb. 24:

Exercises:

127: 44, 45
169: 25 – 29, 44 – 46, 48, 51
177: 6 – 8, 13 – 15, 17 – 20, 26, 27, 29, 30

Fri., Feb. 21:
Read: § 3.2
Exercises:

127: 37 – 39, 43
169: 12 – 14, 21 – 24, 39 – 43
177: 1 – 5, 9 – 12, 16, 21 – 25

Wed., Feb. 19:
Written Assignment No. 1

For each of the following propositions either provide a proof that the proposition is true or provide a counter-example demonstrating that the proposition is false.

1. If the number of elements in a finite group G with identity e is even, show that there is at least one element g in G such that $g \neq e$ but $g * g = e$.
2. If G is a finite group and there is some non-trivial proper subgroup H of G such that for every element g in G one has the relation $gH = Hg$, then for every subgroup H of the given group G one has $gH = Hg$ for every g in G.

Mon., Feb. 17:
President's Day Recess: **no class**

Fri., Feb. 14:
Read: § 3.1
Exercises:

127: 32 – 34, 36
136: 15 – 18
169: 1 – 8, 16 – 20, 32, 33 – 38

Wed., Feb. 12:
Read: § 2.4 through p. 132
Exercises:

115: 11, 14, 15, 22, 23, 26, 31
126: 16, 19 – 24, 26, 27, 29, 30
135: 1 – 4, 7, 8

Mon., Feb. 10:
Read: § 2.3
Exercises:

102: 22, 28 – 32, 36, 45
114: 7, 9, 10, 13
125: 1 – 6, 12, 13, 15

Fri., Feb. 7:
Read: § 2.2
Exercises:

74: 47, 49 – 51
86: 35, 39, 40 – 47, 52, 53, 59
102: 11, 12, 16, 18, 20, 21
114: 1, 2, 5

Wed., Feb. 5:
Read: § 2.1
Exercises:

73: 34, 36, 40, 46
85: 18, 19, 21, 24, 25, 34, 38

101: 1, 3, 5, 6, 7

Mon., Feb. 3:
Read: § 1.5
Exercises:

61: 29, 30, 33
71: 8 – 13, 22, 23, 32
85: 8 – 15, 17

Fri., Jan. 31:
Read: § 1.4
Exercises:

61: 6 – 9, 11 – 18, 23, 26, 27
71: 1 – 7

Wed., Jan. 29:
Read: Read § 1.3
Exercises:

39: 14 – 22
40: 28, 36
48: 8 – 12, 18
61: 1 – 5

Mon., Jan. 27:
Reading: Scan Ch. 0; Read Ch. 1, §§ 1.1 – 1.2
Each of the following exercises may be done quickly if you have paid close attention to the definitions in the reading.
Exercises:

38: 1 – 13
40: 29 – 32
48: 1 – 7

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