Written Assignment No. 5

due Wednesday, December 7, 2005

General Directions: Written assignments should be submitted typeset. What you submit must represent your own work.

Assigned Exercises

1. Decompose the polynomial $t^{12} - 1$ into irreducible factors in the ring $(\mathbf{Z}/5\mathbf{Z})[t]$.
2. Decompose the polynomial $t^8 - 1$ into irreducible factors in the ring $(\mathbf{Z}/2\mathbf{Z})[t]$.
3. Let G denote the ring $\mathbf{Z} + \mathbf{Z}\sqrt{-1}$ of Gaussian integers.
   (a) Find the set G^* of all units in G.
   (b) Find a greatest common divisor in G for 2 and $5 - \sqrt{-1}$.
4. Let R denote the ring $\mathbf{Z}+\mathbf{Z}\sqrt{-5}$. Explain why 14 and $6+2\sqrt{-5}$ have no greatest common divisor in R. *Hint:* Look at the norms of these elements.
5. Let $m \geq 0$ be an integer, and let R denote the ring $\mathbf{Z} + \mathbf{Z}\sqrt{-5}$. Let T_m denote the additive subgroup of R given by
$$T_m \;=\; \mathbf{Z} \cdot 7 + \mathbf{Z} \cdot (m - \sqrt{-5}) \quad .$$
   (a) Find the smallest value of $m \geq 0$ for which T_m is an ideal in R.
   (b) Find a familiar ring that is isomorphic to the quotient ring R/T_m for the value of m obtained in the previous part.