Classical Algebra Written Assignment No. 4

due Monday, November 25, 2002

Directions

Written assignments must be typeset. While it is neither necessary nor desirable to show small details of computation, you must indicate what you are doing and explain any reasoning used. Accuracy is important; with 5 problems in an assignment worth 5 points, there will be no room for partial credit on a problem.

If you are in the writing intensive division of the course, you must complete each written assignment in a satisfactory way. This may require re-submission after an initial evaluation.

The Analogy Between Z and $F[x]$ when F is a field

1. **Greatest Common Divisor of Polynomials over a Finite Field**

   Find the monic greatest common divisor over the finite field $\mathbf{F}_5$ of the two polynomials

   $$x^4 - 1 \quad \text{and} \quad x^4 - 3x^2 + 1 \quad .$$

2. **Bezout's Identity for Polynomials**

   The monic greatest common divisor of the polynomials

   $$f(x) \;=\; x^5 - x + 1 \quad \text{and} \quad g(x) \;=\; x^3 - x + 1 \;,$$

   regarded as polynomials with rational coefficients, is the constant polynomial 1. Express 1 as a polynomial linear combination of f and g. (Be sure to verify the correctness of your answer by expanding the linear combination.)

3. **$(x-2)$-adic Expansion of a Polynomial**

   Expand the polynomial $x^5 - x + 1$ relative to the base $x - 2$ in the ring of polynomials with rational coefficients.

4. **Solving a Polynomial Congruence**

   Determine all polynomials $f(x)$ with rational coefficients for which the polynomial congruence

   $$(x^3 + 2x^2 - x - 2) \cdot f(x) \equiv x^2 + 2x - 3 \mod x^2 - 3x + 2$$

   is satisfied.

5. **$(x-1)$-adic Expansion of a Polynomial Fraction**

   Recall that rational numbers, i.e., integer ratios, have decimal expansions relative to a given base. Find the analogous expansion for the ratio of polynomials (with rational coefficients)

   $$\frac{x}{x^2 - 3x + 3}$$

   relative to $(x - 1)$ as the polynomial base.

   Do the "digits" repeat for this example?