Math 220 Class Slides

http://math.albany.edu/pers/hammond/course/mat220/
Course Assignments Slides

March 13, 2008

1 Reminder

Midterm Test

Tuesday

March 18

2 Matrix of a Linear Map for a Pair of Bases

The transport diagram:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \alpha_{\mathbf{v}} \uparrow & & \uparrow \alpha_{\mathbf{w}} \\ \mathbf{R}^n & \xrightarrow[f]{} & \mathbf{R}^m \end{array}$$

The linear map f between Euclidean spaces has a matrix M

$$f(x) \;=\; f_M(x) \;=\; Mx$$

Definition. M is called the *matrix of ϕ for the pair of bases*

$$\mathbf{v} \;=\; (v_1 v_2 \ldots v_n) \;\text{ and }\; \mathbf{w} \;=\; (w_1 w_2 \ldots w_m) \quad .$$

3 Matrix for a $\pi/2$ Rotation in $\mathbf{R}^3$

- **Question.** If P is the plane in $\mathbf{R}^3$ that is the linear span of the vectors

$$v_1 \;=\; \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \;,\;\; v_2 \;=\; \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \;,$$

and ρ is the rotation in space through $\pi/2$ about the axis through the origin that is perpendicular to P, specify a basis $\mathbf{v} = (v_1 v_2 v_3)$ of $\mathbf{R}^3$ relative to which the matrix of ρ is relatively simple.

- **Answer.** There is slight ambiguity since it is not possible to distinguish between clockwise and counterclockwise.

  (v_1, v_2) is a basis of the plane P

  One computes the "dot product":

  $$v_1 \cdot v_2 \;=\; 1 \cdot 2 + 2 \cdot 1 + (2)(-2) \;=\; 0$$

  So v_1 and v_2 are perpendicular.

  One of the two possible rotations ρ through $\pi/2$ will satisfy:

  $$\rho(v_1) \;=\; v_2 \;\text{ and }\; \rho(v_2) \;=\; -v_1 \quad .$$

  The "cross product" $v_1 \times v_2$ lies on the axis of rotation:

  $$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

  Take $v_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, a vector on the axis, as a third basis vector for $\mathbf{R}^3$.

  $$\rho(v_3) \;=\; v_3$$

  With $\mathbf{v} \;=\; (v_1 v_2 v_3) \;=\; \mathbf{w}$ as selected pairs of bases, the matrix of ρ is:

  $$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

4 Standard Matrix for the $\pi/2$ Rotation

- We have:

  $$\mathbf{v} \;=\; (v_1 v_2 v_3) \;=\; \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \;=\; Q$$

  $$\rho(\mathbf{v}) \;=\; (\rho(v_1)\rho(v_2)\rho(v_3)) \;=\; (v_1 v_2 v_3) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;=\; \mathbf{v}K$$

- **Note:** The second 3×3 matrix K is the matrix of the linear map ρ with respect to the basis pair $\mathbf{w} \;=\; \mathbf{v}$.

  The first 3×3 matrix Q is not the matrix of a linear map but, rather, a matrix whose columns are the standard coordinates — coordinates with respect to the standard basis — of the members of the basis $\mathbf{v}$.

  The matrix corresponding in a similar way to the standard basis $\mathbf{e} \;=\; (e_1 e_2 e_3)$ is the identity matrix, and it would be more precise, instead of writing $\mathbf{v} \;=\; Q$ to use Q to relate the row of vectors $\mathbf{v}$ to the row of vectors $\mathbf{e}$:

  $$\mathbf{v} \;=\; \mathbf{e}Q \quad .$$

  Q is the *matrix for change of basis* between the basis $\mathbf{v}$ and the standard basis $\mathbf{e}$.

- For the standard matrix M of ρ one has $\rho(e_j) = Me_j$ or

$$\rho(\mathbf{e}) = (\rho(e_1)\rho(e_2)\rho(e_3)) = (e_1e_2e_3)M = \mathbf{e}M \quad .$$

- Since ρ is linear, and $\mathbf{v} = \mathbf{e}Q$, one has

$$\rho(\mathbf{v}) = \rho(\mathbf{e}Q) = \rho(\mathbf{e})Q\,,$$

and, therefore, combining the various formulas:

$$\mathbf{e}QK = \mathbf{v}K = \rho(\mathbf{v}) = \rho(\mathbf{e}Q) = \rho(\mathbf{e})Q = \mathbf{e}MQ$$

yielding the following relation among ordinary 3×3 matrices:

$$QK = MQ \text{ or } M = QKQ^{-1}$$

- Because this particular matrix Q consists of mutually perpendicular columns, all of the same length, it is particularly easy to invert:

$$Q^{-1} = (1/9)\,{}^tQ = (1/9)Q = (1/9)\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \quad .$$

- $$M = \frac{1}{9}\begin{pmatrix} 4 & -1 & 8 \\ -7 & 4 & 4 \\ -4 & -8 & 1 \end{pmatrix} \quad .$$

- M is the standard matrix of one of the two rotations through the angle $\pi/2$ about the line through the origin and the point $(2, -2, 1)$.

5 March 11: Exercise No. 1

Let g be the linear map from $\mathbf{R}^4$ to $\mathbf{R}^4$ that is defined by $g(x) = Bx$ where B is the matrix

$$\begin{pmatrix} 1 & 2 & -4 & 3 \\ -2 & -1 & -1 & 5 \\ 1 & 3 & 2 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \quad .$$

Find a 4×4 matrix C for which the linear map h given by multiplication by C has the property that both $h(g(x)) = x$ and $g(h(y)) = y$ for all x and all y in $\mathbf{R}^4$.

- h is the inverse map to g. It is the linear map given by the **inverse matrix**.
- The inverse matrix:

$$\begin{pmatrix} 8/3 & -29/9 & -2/9 & -71/9 \\ -1 & 4/3 & 1/3 & 10/3 \\ 2/3 & -8/9 & 1/9 & -23/9 \\ 1 & -1 & 0 & -3 \end{pmatrix}$$

6 March 11: Exercise No. 2

Let f be a linear map from $\mathbf{R}^3$ to $\mathbf{R}^3$ for which

1. $f(1, 0, 0) = (1, 2, 3)$.
2. $f(0, 1/2, 0) = (3, 2, 1)$.

3. $f(-1,0,2) = (4,-6,2)$.

Find all possible 3×3 matrices A for which the formula $f(x) = Ax$ is valid for all x in $\mathbf{R}^3$.

Hint: Use the rules for abstract linearity to work out what happens under f to $(0,1,0)$ and $(0,0,1)$.

- f is determined by its values on the members of a basis.
- $\{(1,0,0),(0,1/2,0),(-1,0,2)\}$ is a set of 3 linearly independent vectors in $\mathbf{R}^3$, hence, a basis of $\mathbf{R}^3$.
- The columns of A are the values of f on the standard basis.
- $f(0,1,0) = 2f(0,1/2,0) = (6,4,2)$.
- $(0,0,1) = (1/2)\,((-1,0,2)+(1,0,0))$.
- $f(0,0,1) = (1/2)\,((4,-6,2)+(1,2,3)) = (5/2,-2,5/2)$.
- The unique matrix A is
$$\begin{pmatrix} 1 & 6 & 5/2 \\ 2 & 4 & -2 \\ 3 & 2 & 5/2 \end{pmatrix} .$$

7 March 11: Exercise No. 3

For a given real number θ find a 2×2 matrix R_θ for which the linear function ρ defined by $\rho(x) = R_\theta x$ is the counterclockwise rotation of the plane through the angle of (radian) measure θ.

- $$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$
- e_2 "ahead" of e_1 by $\pi/2$
- $$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \cos(\theta+\pi/2) \\ \sin(\theta+\pi/2) \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$
- $$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

8 March 11: Exercise No. 4

Find a 3×3 matrix S for which the linear function σ given by $\sigma(x) = Sx$ is the reflection of $\mathbf{R}^3$ in the xz plane (where the 2^{nd} coordinate $y = 0$).

- Points in the xz plane do not move.
- Points on the y-axis are "flipped", i.e., $(0,y,0) \mapsto (0,-y,0)$.
- $$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9 March 13: Exercise No. 1

When M is an $m \times n$ matrix, the phrase "corresponding linear function" will denote the linear function

$$\mathbf{R}^n \xrightarrow{f_M} \mathbf{R}^m$$

defined by

$$f_M(x) \;=\; Mx \;\; \text{for} \;\; x \text{ in } \mathbf{R}^n \quad .$$

In the case $m = 2, \; n = 3$

$$M \;=\; \begin{pmatrix} 3 & 6 & 0 \\ 2 & 4 & 1 \end{pmatrix}$$

compute each of the following items both for (i) M itself and for (ii) its reduced row echelon form:

a. The set of linear combinations of the columns.
b. The set of linear combinations of the rows.
c. The set of linear relations among the columns.
d. The set of linear relations among the rows.
e. The kernel of the corresponding linear function.
f. The image of the corresponding linear function.

The reduced row echelon form is

$$R \;=\; \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

a. Since the two rows are not parallel, M and R have rank 2. Hence, the dimension of both column spaces is 2, and both column spaces are simply $\mathbf{R}^2$.
b.

(1) Row space does not change under row operations.
(2) Basis of both row spaces:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(3) Equation characterizing the common row space as a subspace of $\mathbf{R}^3$:

$$x_2 - 2x_1 \;=\; 0$$

c. Linear relations among the columns are the same for M and R. Non-redundant characterizing linear relations are obtained by expressing each non-pivot column in terms of the pivot columns:

$$2C_1 - C_2 \;=\; 0$$

d. There are only 2 rows, which are linearly independent. Thus, **no linear relations among the rows**.
e. The kernel is the same for both M and R.

$$\dim(\text{Kernel}) \;=\; \dim(\text{domain}) - \dim(\text{Image}) \;=\; 3 - 2 \;=\; 1$$

A basis for the common kernel:

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

f. In each case the image is the column space. (For a general matrix its column space will differ from the column space of its RREF.) In both cases the image is $\mathbf{R}^2$.

10 March 13: Exercise No. 2

Let Q_3 be the 4-dimensional vector space consisting of all polynomials of degree 3 or less, and let

$$\mathbf{v} \;=\; \{1, t, t^2, t^3\}$$

be the familiar basis of Q_3 . Let $Q_3 \xrightarrow{\phi} Q_3$ be the linear map that is defined by

$$\phi(P) \;=\; P'' + 3P' + 2P \ ,$$

where P' and P'' denote the first and second derivatives of P . Find the matrix of ϕ with respect to the basis $\mathbf{v}$, i.e., find the 4×4 matrix R that appears in the transport diagram

$$\begin{array}{ccc} Q_3 & \xrightarrow{\phi} & Q_3 \\ \alpha_{\mathbf{v}} \uparrow & & \uparrow \alpha_{\mathbf{v}} \\ \mathbf{R}^4 & \xrightarrow{f_M} & \mathbf{R}^4 \end{array} \quad .$$

- Compute the 4 polynomials $\phi(t^j)$ for $0 \leq j \leq 3$.
- The 4 columns of M are the coefficient vectors for these polynomials.