Math 220 Class Slides

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1 Effect of Row Operations on a Matrix: I

Row Space Unchanged

Proposition. For a given matrix each of the three kinds of elementary row operations leaves the row space of the matrix unchanged.

2 Effect of Row Operations on a Matrix: II

Linear Relations Among Columns Unchanged

Proposition. For a given matrix each of the three kinds of elementary row operations leaves the set of linear relations among the columns unchanged.

3 Linearly Independent Vectors

Let V be any vector space.

Definition. A sequence $v_1, v_2, \ldots, v_r$ of elements of V is *linearly independent* if no non-trivial linear combination of $v_1, v_2, \ldots, v_r$ vanishes.

Re-stated:

$v_1, v_2, \ldots, v_r$ are linearly independent if and only if the only solution of

$$c_1 v_1 + c_2 v_2 + \ldots c_r v_r \;=\; \vec{0}$$

is given by

$$c_1 \;=\; c_2 \;=\; \ldots \;=\; c_r \;=\; 0$$

Definition. A sequence $v_1, v_2, \ldots, v_r$ of elements of V is *linearly dependent* if it is not linearly independent.

4 Finite Dimensional Vector Spaces

Definition. A vector space V is *finite-dimensional* (or *finitely spanned* or *finitely generated*) if there is a finite sequence of elements $v_1, v_2, \ldots, v_r$ in V such that V is the linear span of $v_1, v_2, \ldots, v_r$.

This means that each v in V is a linear combination of $v_1, v_2, \ldots, v_r$.

Example. $\mathbf{R}^n$ is finite-dimensional since it is spanned by

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} .$$

5 A Fundamental Inequality

Proposition. In any finite-dimensional vector space the number of elements in any linearly independent sequence is at most equal to the number of elements in a given spanning set.

Proof. Let the vector space be spanned by $w_1, \ldots, w_m$, and let $v_1, \ldots v_n$ be a linearly independent sequence. The task is to show $n \leq m$.

Since $w_1, \ldots, w_m$ is a spanning set, one has

$$v_j \;=\; a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{mj}w_m$$

for each j, $1 \leq j \leq n$. One may express this very concisely by writing

$$v \;=\; wA$$

where v is the $1 \times n$ row $(v_1 v_2 \ldots v_n)$ of elements of V, w is the $1 \times m$ row $(w_1 w_2 \ldots w_m)$ of elements of V, and A is the $m \times n$ matrix $A \;=\; (a_{ij})$.

If $n > m$, then the reduced row echelon form of A can have at most m non-zero rows and, therefore, at most m pivot columns. So at least one column of A is not a pivot column. This means that column is a linear combination of the pivot columns to its left. Hence, if x is the column of n coefficients of the ensuing linear relation $Ax \;=\; 0$ with $x \neq 0$, then one has

$$vx \;=\; (wA)x \;=\; w(Ax) \;=\; w\,0 \;=\; 0\;,$$

which means that $v_1, \ldots v_n$ cannot be linearly independent, a contradiction made possible by assuming $n > m$. Hence $n \leq m$.

6 Basis of a Vector Space

Definition. A *basis* of a vector space V is any maximal linearly independent subset of the vector space.

Here the word *maximal* indicates a linearly independent set that is not a subset of a (strictly) larger linearly independent set.

Proposition. A subset of a vector space V is a basis if and only if it is a linearly independent set and it spans V.

Proof. Certainly a linearly independent spanning set must be a maximal linearly independent set. Conversely if S is a maximal linearly independent set, and v is any element of V, then the set $S \cup \{v\}$ cannot be linearly independent by the maximality of S. Hence, there must be a non-trivial linear relation among the members of a finite subset of $S \cup \{v\}$. The element v must be involved with non-zero coefficient in that linear relation since there can be no such relation among finitely many members of S. That linear relation can be used to obtain v as a linear combination of finitely many members of S. Therefore v, which was an arbitary member of V, lies in the span of S.

7 Dimension of a Vector Space

Theorem. In a finite dimensional vector space any two bases have the same number of elements.

Proof. Apply the fundamental inequality twice.

Definition. The *dimension* of a finite dimensional vector space is the number of elements in any basis.

Example. $\mathbf{R}^n$ has dimension n.

8 The Case of $\dim(V)$ Spanning Vectors in V

Proposition. If $\dim(V) = n$, and $v_1, \dots v_n$ span V, then $v_1, \dots v_n$ must be linearly independent, i.e., form a basis of V.

Proof. Since $\dim(V) = n$, V must have a basis $w_1, \dots w_n$ with n members. Since $w_1, \dots w_n$ are linearly independent, the fundamental inequality guarantees that n is less than or equal to the number of elements in any spanning set, including a maximal linearly independent subset, also a spanning set, of $v_1, \dots v_n$. Hence, $v_1, \dots v_n$ must be a maximal linearly independent subset of itself, i.e., must be a linearly independent set.

9 The Case of $\dim(V)$ Independent Vectors in V

Proposition. If $\dim(V) = n$, and $v_1, \ldots v_n$ form a linearly independent subset of V, then $v_1, \ldots v_n$ must span V, i.e., form a basis of V.

Proof. Since $\dim(V) = n$, V must have a basis $w_1, \ldots w_n$ with n members. Since $w_1, \ldots w_n$ span V, the fundamental inequality guarantees that a maximal linearly independent subset of V containing $v_1, \ldots, v_n$ must have no more than n elements, i.e., $v_1, \ldots, v_n$ must be a basis since a maximal linearly independent subset of V is, by definition, a basis of V.

10 Dimension of Row and Column Spaces

Theorem. For any $m \times n$ matrix M the dimension of the row space of M is equal to the dimension of the column space of M.

Proof. The dimension of the column space is the number of columns in a maximal linearly independent set of columns. Since linear relations among the columns do not change under row operations, the dimension of the column space is the number of columns in a maximal linearly independent set of columns for the reduced row echelon form. Hence, the dimension of the column space is the number of columns in the reduced row echelon form containing leading 1's.

On the other hand, the row space of any $m \times n$ matrix is the same linear subspace of $\mathbf{R}^m$ as the row space of its reduced row echelon form. It is obvious that a basis of the row space of a matrix in reduced row echelon form is given by the set of its non-zero rows.

Since for a matrix in reduced row echelon form, the number of non-zero rows is equal to the number of columns containing leading 1's, the dimension of the row space of a given matrix is equal to the dimension of its column space.

11 The Rank of a Matrix

Definition. The rank of an $m \times n$ matrix is the number that is both the dimension of its row space and the dimension of its column space.

12 Parametric Representations, Coordinates, and Bases

- To give a parametric representation of a linear subspace in a vector space is to represent a general member of the subspace as a linear combination of the vectors in some basis of the subspace.
- The coefficients of the basis in such a representation are "coordinates" in the linear subspace relative to the basis.
- To have coordinates for the points of a linear subspace of dimension k is to have a linear way of matching points in the subspace with points in $\mathbf{R}^k$.
- To have coordinates for the points of a linear subspace of dimension k is to have an isomorphism from $\mathbf{R}^k$ to the subspace.

13 Linearity

Recall:

Definition. A map $V \xrightarrow{\phi} W$ from a vector space V is a *linear map* if it preserves linear combinations. That is:

For any $v_1, \ldots, v_r$ in V and any scalars $x_1, \ldots, x_r$ one has

$$\phi(x_1 v_1 + \ldots x_r v_r) \;=\; x_1 \phi(x_1) + \ldots + x_r \phi(v_r) \quad .$$

Example:

$$M \text{ an } m \times n \text{ matrix}$$

$$\mathbf{R}^n \xrightarrow{f_M} \mathbf{R}^m$$

$$f_M(x) \;=\; Mx$$

$$f_M \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \;=\; x_1 M_1 + x_2 M_2 + \ldots x_n M_n \quad \text{where } M_j \;=\; j\text{th column of } M$$

14 Linearity in the Euclidean Case

Theorem. For any linear map $\mathbf{R}^n \xrightarrow{\phi} \mathbf{R}^m$ between Euclidean spaces there is a unique $m \times n$ matrix M such that $\phi \;=\; f_M$.

Re-stated: Every linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$ is given in the usual way by some matrix.

Proof. Let

$$e_1 \;=\; \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; e_2 \;=\; \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; \ldots, \; e_n \;=\; \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

— the "standard basis" of $\mathbf{R}^n$.

For any matrix M:

$$Me_j \;=\; M_j \;=\; j\text{th column of } M \text{ for } 1 \le j \le n \quad .$$

For any x in $\mathbf{R}^n$:

$$x \;=\; \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \;=\; x_1e_1 + x_2e_2 + \ldots + x_ne_n \quad .$$

If ϕ is linear, then

$$\phi(x) \;=\; x_1\phi(e_1) + x_2\phi(e_2) + \ldots + x_n\phi(e_n) \quad .$$

Given a linear map ϕ choose M to be the unique matrix with

$$M_j \;=\; \phi(e_j) \text{ for } 1 \le j \le n \quad .$$

Then for all x in $\mathbf{R}^n$:

$$\phi(x) \;=\; x_1M_1 + \ldots + x_nM_n \;=\; Mx \;=\; f_M(x) \quad .$$

Therefore,

$$\phi \;=\; f_M \quad .$$

15 A Fundamental Theorem

Theorem. If $V \xrightarrow{\phi} W$ is a linear map between vector spaces with V finite-dimensional, then

$$\dim(V) \;=\; \dim(\text{Kernel}\phi) + \dim(\text{Image}\phi)$$

Proof in the Euclidean Case:

Assume $V = \mathbf{R}^n$, $W = \mathbf{R}^m$, and $\phi = f_M$ where M is an $m \times n$ matrix.

Recall that the image of f_M is spanned by the columns of M. Thus, the image of f_M is the same subspace of $\mathbf{R}^m$ as the column space of M. Hence,

$$\dim(\text{Image}f_M) \;=\; r \;=\; \text{Rank}(M) \quad .$$

Recall that the kernel of f_M is the subspace of $\mathbf{R}^n$ consisting of all x in $\mathbf{R}^n$ such that $Mx = 0$. When the system of linear equations represented by the matrix equation $Mx = 0$ is put in reduced row echelon form, one is able to solve for the pivot column variables in terms of the other variables. Thus, the $n-r$ non-pivot column variables may be used as parameters for the space of solutions, i.e., for the kernel of f_M. The "space of parameters" has a "standard basis" consisting of $n-r$ value sets for these parameters. Hence,

$$\dim(\text{Kernel}f_M) \;=\; n-r \;=\; \dim(V) - \dim(\text{Image}f_M) \quad .$$

Proof for the general case will be given later by reducing it to this special case.

16 Exercise 1(c)

Which sets of column indices correspond to maximal linearly independent sets of columns in the following matrix?

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Response:

Since $Mx = x_1 M_1 + x_2 M_2 + x_3 M_3$, the question of linear relations among the columns is equivalent to the question of finding solutions of the linear system $Mx = 0$. Compute the reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

With the reduced row echelon form the non-pivot columns are transparently expressed in terms of the pivot columns:

$$M_3 = -M_1 + 2M_2$$

or

$$M_2 = \frac{1}{2}(M_1 + M_3) \quad .$$

Conclusion: Any two of the three columns form a maximal linearly independent set.

Note: In this case the three rows are related in the same way. The fact that the row relations are the same is a special fact in this exercise.

17 Exercise 2(a)

Which sets of row indices correspond to maximal linearly independent sets of rows in the following matrix?

$$N = \begin{pmatrix} 1 & 2 & -4 & 7 \\ -2 & -1 & -1 & -8 \\ -1 & -4 & -14 & 5 \\ 5 & 7 & -11 & 29 \end{pmatrix}$$

Response: The first of two approaches:

Exchange columns and rows. Then follow the method used for exercise 1. The matrix obtained by exchanging columns and rows in a given matrix is called the *transpose* of the given matrix.

The transpose:

$${}^tN = \begin{pmatrix} 1 & -2 & -1 & 5 \\ 2 & -1 & -4 & 7 \\ -4 & -1 & -14 & -11 \\ 7 & -8 & 5 & 29 \end{pmatrix}$$

The RREF of the transpose:

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Column relations in the tranpose give row relations in the original:

$$N^4 = 3N^1 - N^2 \quad .$$

Conclusion: A maximal linearly independent set of the rows of N is given by taking any two of rows 1, 2, and 4 along with row 3.

18 Exercise 2(a) (continued)

Which sets of row indices correspond to maximal linearly independent sets of rows in the following matrix?

$$N = \begin{pmatrix} 1 & 2 & -4 & 7 \\ -2 & -1 & -1 & -8 \\ -1 & -4 & -14 & 5 \\ 5 & 7 & -11 & 29 \end{pmatrix}$$

Response: The second of two approaches:

Linear relations among the rows of N correspond to equations satisfied by points y in the image of the linear map

$$\mathbf{R}^4 \xrightarrow{f_N} \mathbf{R}^4$$

Form the generic augmented matrix

$$N = \begin{pmatrix} 1 & 2 & -4 & 7 & y_1 \\ -2 & -1 & -1 & -8 & y_2 \\ -1 & -4 & -14 & 5 & y_3 \\ 5 & 7 & -11 & 29 & y_4 \end{pmatrix} .$$

Use row operations to maneuver this augmented matrix so that the coefficient matrix portion is in reduced row echelon form, and then extract the linear relations among the coordinates of y arising from zero rows in the coefficient portion.

$$y_4 - 3y_1 + y_2$$