Math 220 Class Slides

http://math.albany.edu/pers/hammond/course/mat220/

February 28, 2008

1 Linear Combinations and *Span*

Definition. If V is a vector space and $v_1, v_2, \ldots, v_r$ are elements of V (vectors), then a *linear combination of* $v_1, v_2, \ldots, v_r$ is an element of V having the form $c_1 v_1 + c_2 v_2 + \ldots + c_r v_r$ for some scalars $c_1, c_2, \ldots, c_r$.

Proposition. The set of all linear combinations of $v_1, v_2, \ldots, v_r$ is a linear subspace of V.

The proof is obvious.

Definition. The set of all linear combinations of $v_1, v_2, \ldots, v_r$ is called the *linear span of* $v_1, v_2, \ldots, v_r$ or may also be called the *linear subspace of* V *generated by* $v_1, v_2, \ldots, v_r$.

2 The Row and Column Spaces of a Matrix

Suppose M is an $m \times n$ matrix with

columns $M_1, M_2, \ldots M_n$
rows $M^1, M^2, \ldots M^m$ (superscripts)

Definition.

The *column space* of M is the linear span of $M_1, M_2, \ldots M_n$.

The *row space* of M is the linear span of $M^1, M^2, \ldots M^m$.

Proposition. If $\mathbf{R}^n \xrightarrow{f_M} \mathbf{R}^m$ is the linear map given by $f_M(x) = Mx$, then the image of f_M is the column space of M.

Proof. The nature of matrix multiplication is such that

$$Mx = x_1 M_1 + x_2 M_2 + \ldots x_n M_n \quad .$$

3 Effect of Row Operations on a Matrix: I

Row Space Unchanged

Proposition. For a given matrix each of the three kinds of elementary row operations leaves the row space of the matrix unchanged.

Proof. Use case by case checking.

4 Effect of Row Operations on a Matrix: II

Linear Relations Among Columns Unchanged

Proposition. For a given matrix each of the three kinds of elementary row operations leaves the set of linear relations among the columns unchanged.

Proof. A linear relation among the columns of M is a relation, if true, of the form

$$a_1 M_1 + \ldots + a_n M_n \ = \ b_1 M_1 + \ldots + b_n M_n$$

for scalars $a_1, \ldots, a_n, b_1, \ldots, b_n$. Such a relation holds if and only if

$$(a_1 - b_1) M_1 + \ldots (a_n - b_n) M_n \ = \ \vec{0} \quad .$$

This holds if and only if the column $a - b$ is a solution of the linear system of equations

$$Mx \ = \ \vec{0}$$

by application of the relation

$$Mx \ = \ x_1 M_1 + \ldots + x_n M_n$$

to the case $x \ = \ a - b$.

5 Linearly Independent Vectors

Let V be any vector space.

Definition. A sequence $v_1, v_2, \ldots, v_r$ of elements of V is *linearly independent* if no non-trivial linear combination of $v_1, v_2, \ldots, v_r$ vanishes.

Re-stated:

$v_1, v_2, \ldots, v_r$ are linearly independent if and only if the only solution of

$$c_1 v_1 + c_2 v_2 + \ldots c_r v_r \ = \ \vec{0}$$

is given by

$$c_1 \ = \ c_2 \ = \ \ldots \ = \ c_r \ = \ 0$$

Definition. A sequence $v_1, v_2, \ldots, v_r$ of elements of V is *linearly dependent* if it is not linearly independent.

6 Example of Linear Independence

In $\mathbf{R}^n$ the unit vectors on the n positive coordinate axes

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

are linearly independent.

7 Example of Linear Dependence

Example. In $\mathbf{R}^2$ the 3 vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

are linearly dependent since

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad .$$

8 Another Example

Proposition. The columns M_1, M_2, $\ldots$ M_n of an $m \times n$ matrix are linearly independent if and only if the only solution of the linear system $Mx = 0$ is the solution $x = 0$.

Re-statement: For an $m \times n$ matrix M the columns M_1, M_2, $\ldots$ M_n are linearly independent if and only if the kernel of the linear map

$$\mathbf{R}^n \xrightarrow{f_M} \mathbf{R}^m$$

consists only of the vector 0.

And again: For an $m \times n$ matrix M the columns M_1, M_2, $\ldots$ M_n are linearly dependent if and only if there is a vector x in $\mathbf{R}^n$ such that $x \neq 0$ yet $Mx = 0$.

9 The Uniqueness of Reduced Row Echelon Form

Proposition. There is only one reduced row echelon form that may be obtained from a given matrix.

Proof. This boils down to the question of whether, for matrices of given size, a matrix in reduced row echelon form is completely characterized by the linear relations among its columns. This is seen to be true as follows:

1. The indices corresponding to pivot columns (columns containing leading 1's in reduced row echelon form) are the indices of the "left-most" maximal linearly independent subset the of set of columns.
2. In reduced row echelon form each non-pivot column is a linear combination, in a unique way, of the pivot columns to its left.

10 Finite Dimensional Vector Spaces

Definition. A vector space V is *finite-dimensional* (or *finitely spanned* or *finitely generated*) if there is a finite sequence of elements $v_1, v_2, \ldots, v_r$ in V such that V is the linear span of $v_1, v_2, \ldots, v_r$.

This means that each v in V is a linear combination of $v_1, v_2, \ldots, v_r$.

Example. $\mathbf{R}^n$ is finite-dimensional since it is spanned by

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad .$$

11 Infinite Linearly Independent Sets

Definition. If V is a vector space, a subset S, finite or infinite, of V is *linearly independent* if each finite sequence of distinct members of S is linearly independent.

Example. Let P be the vector space of all polynomials in the variable t. Then the set

$$S = \{1, t, t^2, t^3, \ldots, t^k, \ldots\}$$

of all powers of t is a linearly independent set. (This may be proved easily using Taylor's Theorem.)

12 Infinite Spanning Sets

Definition. If V is a vector space, a subset S, finite or infinite, of V *spans* or *generates* V if each member of V is a linear combination of the members of a finite sequence in S.

Example. Let P be the vector space of all polynomials in the variable t. Then the set

$$S = \{1, t, t^2, t^3, \ldots, t^k, \ldots\}$$

of all powers of t spans P (by the definition of *polynomial*).

13 A Fundamental Inequality

Proposition. In any finite-dimensional vector space the number of elements in any linearly independent sequence is at most equal to the number of elements in a given spanning set.

Proof. Let the vector space be spanned by $w_1, \ldots, w_m$, and let $v_1, \ldots v_n$ be a linearly independent sequence. The task is to show $n \leq m$.

Since $w_1, \ldots, w_m$ is a spanning set, one has

$$v_j = a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{mj}w_m$$

for each j, $1 \leq j \leq n$. One may express this very concisely by writing

$$v = wA$$

where v is the $1 \times n$ row $(v_1 v_2 \ldots v_n)$ of elements of V, w is the $1 \times m$ row $(w_1 w_2 \ldots w_m)$ of elements of V, and A is the $m \times n$ matrix $A = (a_{ij})$.

If $n > m$, then the reduced row echelon form of A can have at most m non-zero rows and, therefore, at most m pivot columns. So at least one column of A is not a pivot column. This means that column is a linear combination of the pivot columns to its left. Hence, if x is the column of n coefficients of the ensuing linear relation $Ax = 0$ with $x \neq 0$, then one has

$$vx = (wA)x = w(Ax) = w\,0 = 0\ ,$$

which means that $v_1, \ldots v_n$ cannot be linearly independent, a contradiction made possible by assuming $n > m$. Hence $n \leq m$.

14 Example of an Infinite Dimensional Vector Space

The vector space of all polynomials (of all degrees) in the variable t is **not** a finite dimensional vector space because it contains the infinite linearly independent set $\{1, t, t^2, \ldots\}$ of all powers of t.

15 Basis of a Vector Space

Definition. A *basis* of a vector space V is any maximal linearly independent subset of the vector space.

Here the word *maximal* indicates a linearly independent set that is not a subset of a (strictly) larger linearly independent set.

Proposition. A subset of a vector space V is a basis if and only if it is a linearly independent set and it spans V.

Proof. Certainly a linearly independent spanning set must be a maximal linearly independent set. Conversely if S is a maximal linearly independent set, and v is any element of V, then the set $S \cup \{v\}$ cannot be linearly independent by the maximality of S. Hence, there must be a non-trivial linear relation among the members of a finite subset of $S \cup \{v\}$. The element v must be involved with non-zero coefficient in that linear relation since there can be no such relation among finitely many members of S. That linear relation can be used to obtain v as a linear combination of finitely many members of S. Therefore v, which was an arbitary member of V, lies in the span of S.

16 Examples of Bases

- The n unit vectors on the positive coordinate axes form a basis of $\mathbf{R}^n$.
- The (infinite) set of all powers of t forms a basis of the space of all polynomials in the variable t.

17 Dimension of a Vector Space

Theorem. In a finite dimensional vector space any two bases have the same number of elements.

Proof. Apply the fundamental inequality twice.

Definition. The *dimension* of a finite dimensional vector space is the number of elements in any basis.

Example. $\mathbf{R}^n$ has dimension n.

18 Assignment: Exercise No. 2

Let f be the linear map from $\mathbf{R}^4$ to $\mathbf{R}^4$ that is given by the matrix

$$\begin{pmatrix} 1 & 2 & -4 & 7 \\ -2 & -1 & -1 & -8 \\ -1 & 4 & -14 & 5 \\ 5 & 7 & -11 & 29 \end{pmatrix} .$$

a. Obtain a parametric representation for the kernel of f.
b. Find a pair of equations in 4 variables that characterize the image of f.
c. List a pair of equations in 4 variables that characterize the kernel of f.
d. Give a parametric representation for the image of f.

19 RREF of the Generic Augmented Matrix

$$\begin{pmatrix} 1 & 2 & -4 & 7 & y_1 \\ -2 & -1 & -1 & -8 & y_2 \\ -1 & 4 & -14 & 5 & y_3 \\ 5 & 7 & -11 & 29 & y_4 \end{pmatrix}$$

$$\xrightarrow{\text{row ops}}$$

$$\begin{pmatrix} 1 & 0 & 2 & 3 & -(y_1 + 2y_2)/3 \\ 0 & 1 & -3 & 2 & (2y_1 + y_2)/3 \\ 0 & 0 & 0 & 0 & y_3 - 3y_1 - 2y_2 \\ 0 & 0 & 0 & 0 & y_4 - 3y_1 + y_2 \end{pmatrix}$$

20 Part (a): Parametric Representation of the Kernel

- $y_1 = y_2 = y_3 = y_4 = 0$.
- Columns 1 and 2 are pivot columns.
- Equations may be solved for x_1 and x_2 in terms of x_3 and x_4.
- The two non-trivial equations:

$$\begin{cases} x_1 = -2x_3 - 3x_4 \\ x_2 = 3x_3 - 2x_4 \end{cases}$$

- Let $u = x_3$ and $v = x_4$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = u \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} + v \begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

21 Part (a): Observations

- The parametric representation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = u \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} + v \begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

- Two parameters u and v.
- The kernel is a plane through 0 in $\mathbf{R}^4$.
- The kernel is a linear subspace of $\mathbf{R}^4$.
- A basis of the kernel.

$$\left\{ \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

- The kernel has dimension 2.
- The linear map $\mathbf{R}^2 \xrightarrow{\phi} \mathbf{R}^4$

$$\phi(u, v) = u \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} + v \begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

  is an isomorphism from $\mathbf{R}^2$ to the kernel.
- u and v are "coordinates" for points in the kernel relative to this basis.

22 Part (b): Equations for the Image

- Want equations in the y_j.
- Look at the rows of the reduced augmented matrix with zeros in the "coefficient" columns.
- Use the corresponding linear equations.
- $$\left\{ \begin{array}{rcl} y_3 - 3y_1 - 2y_2 & = & 0 \\ y_4 - 3y_1 + y_2 & = & 0 \end{array} \right.$$

23 Part (c): Equations for the Kernel

- Want equations in the x_i.
- Look at the rows of the reduced augmented matrix that are non-zero in the "coefficient" columns.
-

$$\begin{cases} x_1 + 2x_3 + 3x_4 & = & 0 \\ x_2 - 3x_3 + 2x_4 & = & 0 \end{cases}$$

- This is easier than part(a).

24 Part (d): Parametric Representation for the Image

- A basis for the image is given by the pivot columns in the **original matrix**.
- The pivot columns are the first and second:

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \\ 7 \end{pmatrix} \right\}$$

- Parametric Representation:

$$\psi(s,t) \;=\; s \begin{pmatrix} 1 \\ -2 \\ -1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 4 \\ 7 \end{pmatrix}$$

- Easier than part(b).
- The linear map $\mathbf{R}^2 \xrightarrow{\psi} \mathbf{R}^4$ is an isomorphism from $\mathbf{R}^2$ to the image.
- s and t are "coordinates" for points in the image relative to this basis.

25 Parametric Representations, Coordinates, and Bases

- To give a parametric representation of a linear subspace in a vector space is to represent a general member of the subspace as a linear combination of the vectors in some basis of the subspace.
- The coefficients of the basis in such a representation are "coordinates" in the linear subspace relative to the basis.
- To have coordinates for the points of a linear subspace of dimension k is to have a linear way of matching points in the subspace with points in $\mathbf{R}^k$.
- To have coordinates for the points of a linear subspace of dimension k is to have an isomorphism from $\mathbf{R}^k$ to the subspace.