Math 220 Class Slides

http://math.albany.edu/pers/hammond/course/mat220/

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1 The Last Quiz

Where does the line L through $A = (4, 2, -7)$ and $B = (10, 5, 8)$ meet the plane $x - y + z = 7$?

Parametric representation of L:

$$P = \phi(t) = (1-t)A + tB = (4 + 6t, 2 + 3t, -7 + 15t)$$

When does $\phi(t)$ satisfy the equation of the plane?

$$x - y + z = (4 + 6t) - (2 + 3t) + (-7 + 15t) = -5 + 18t = 7$$

Solve for t:

$$t = \frac{2}{3}$$

$$\boxed{P = (8, 4, 3)}$$

2 The Last Quiz Again

Where does the line L through $A = (4, 2, -7)$ and $B = (10, 5, 8)$ meet the plane $x - y + z = 7$?

What Does Not Help

The matrix

$$\begin{pmatrix} 4 & 2 & -7 & 0 \\ 10 & 5 & 8 & 0 \\ 1 & -1 & 1 & 7 \end{pmatrix}$$

is the augmented matrix of the linear system

$$\begin{cases} 4x + 2y - 7z & = & 0 \\ 10x + 5y + 8z & = & 0 \\ x - y + z & = & 7 \end{cases}$$

This linear system does not bear on the problem.

3 Exercise No. 1

Let C be the 4×4 matrix

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ -2 & -1 & 3 & 2 \\ -2 & 2 & 6 & -1 \\ 1 & 0 & -2 & 0 \end{pmatrix},$$

and let f be the linear map (or function) from $\mathbf{R}^4$ to $\mathbf{R}^4$ defined by the formula

$$y = f(x) = Cx \quad .$$

a. Find all solutions of $f(x) = (0,0,0,0)$.

b. Find all solutions of $f(x) = (1,-2,-2,1)$ with $x_3 = 0$.

c. Find all solutions of $f(x) = (1,-2,-2,1)$.

d. Find all solutions of $f(x) = (-1,-7,2,1)$ with $x_3 = 0$.

e. Find all solutions of $f(x) = (-1,-7,2,1)$.

f. What is the kernel of f?

g. Find equations that characterize the image of f.

4 Exercise No. 1: Augmented Matrix

$$\begin{pmatrix} 1 & 2 & 0 & 2 & y_1 \\ -2 & -1 & 3 & 2 & y_2 \\ -2 & 2 & 6 & -1 & y_3 \\ 1 & 0 & -2 & 0 & y_4 \end{pmatrix}$$

Use row operations to bring the first 4 columns into RREF.

$$\begin{pmatrix} 1 & 0 & -2 & 0 & (y_1 - 2y_2 - 2y_3)/9 \\ 0 & 1 & 1 & 0 & (2y_1 - y_2 + 2y_3)/9 \\ 0 & 0 & 0 & 1 & (2y_1 + 2y_2 - y_3)/9 \\ 0 & 0 & 0 & 0 & (9y_4 - y_1 + 2y_2 + 2y_3)/9 \end{pmatrix}$$

5 Exercise No. 1: Part (g)

- The **image** of f is the set of all y for which the fiber of f over y is non-empty, or, equivalently, the set of all y for which the equation $y = f(x)$ has at least one solution x.
- Equations corresponding to the image of a linear map given by a matrix are obtained from rows in the augmented matrix for which the coefficient matrix portion of the row is zero.
- In this case

$$9y_4 - y_1 + 2y_2 + 2y_3 = 0$$

6 Exercise No. 1: Parts (a) & (f)

- The **kernel** of f is the set of solutions of $f(x) = 0$.
- The equations:

$$\begin{cases} x_1 - 2x_3 & = & 0 \\ x_2 + x_3 & = & 0 \\ x_4 & = & 0 \\ 0 & = & 0 \end{cases}$$

- Variables corresponding to pivot columns — x_1, x_2, and x_4 — may be expressed in terms of the others — x_3:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

- The kernel may be described as:

  The line in $\mathbf{R}^4$ through the origin and the point $(2, -1, 1, 0)$
  OR
  The linear subspace of $\mathbf{R}^4$ consisting of all scalar multiples of $(2, -1, 1, 0)$

7 Exercise No. 1: the specific equations

- Solution for $y = (1, -2, -2, 1)$:

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

  $f^{-1}(1, -2, -2, 1)$ is the translate of $\mathrm{Ker}(f)$ by the vector $(1, 0, 0, 0)$.

- $(1, 0, 0, 0)$ is the solution for which $x_3 = 0$
- Solution for $y = (-1, -7, 2, 1)$:

$$x = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

  $f^{-1}(-1, -7, 2, 1)$ is the translate of $\mathrm{Ker}(f)$ by the vector $(1, 1, 0, -2)$.

- $((1, 1, 0, -2))$ is the solution for which $x_3 = 0$

8 Exercise No. 2

Let G be the 4×4 matrix

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ -2 & -1 & 1 & 1 \\ -1 & 4 & 2 & 5 \\ 5 & 7 & -1 & 2 \end{pmatrix},$$

and let g be the linear map (or function) from $\mathbf{R}^4$ to $\mathbf{R}^4$ defined by the formula

$$y = g(x) = Gx \quad .$$

Solve each of the following systems of 4 linear equations in 4 unknowns x_1, x_2, x_3 and x_4.

a. $g(x) = (0, 0, 0, 0)$.

b. $g(x) = (1, -1, 1, 3)$ with $x_3 = 0$.

c. $g(x) = (1, -1, 1, 4)$ with $x_3 = 0$.

d. $g(x) = (1, -1, 1, 4)$ with $x_3 = x_4 = 0$.

e. $g(x) = (3, -1, 2, 1)$ with $x_3 = 0$.

f. $g(x) = (3, -1, 7, 10)$ with $x_3 = 0$.

g. What is the kernel of g?

h. Find equations that characterize the image of f.

9 Exercise No. 2: the Augmented Matrix

$$\begin{pmatrix} 1 & 2 & 0 & 1 & y_1 \\ -2 & -1 & 1 & 1 & y_2 \\ -1 & 4 & 2 & 5 & y_3 \\ 5 & 7 & -1 & 2 & y_4 \end{pmatrix}$$

Use row operations to bring the first 4 columns into RREF.

$$\begin{pmatrix} 1 & 0 & -2/3 & -1 & -(y_1 + 2y_2)/3 \\ 0 & 1 & 1/3 & 1 & (2y_1 + y_2)/3 \\ 0 & 0 & 0 & 0 & y_3 - 3y_1 - 2y_2 \\ 0 & 0 & 0 & 0 & y_4 - 3y_1 + y_2 \end{pmatrix}$$

10 Exercise No. 2: Kernel and Image

- **Image**:
$$y_3 \;=\; 3y_1 + 2y_2 \quad \text{and} \quad y_4 \;=\; 3y_1 - y_2$$
- **Note:** Each column of the original matrix G is in the image.
- **Kernel** given by equations:
$$x_1 \;=\; (2/3)x_3 + x_4 \qquad x_2 \;=\; -(1/3)x_3 - x_4$$
- **Kernel** in parametric form (with $u \;=\; x_3$ and $v \;=\; x_4$):
$$x \;=\; u \begin{pmatrix} 2/3 \\ -1/3 \\ 1 \\ 0 \end{pmatrix} + v \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$
- Every vector in the kernel is a linear combination of
$$\begin{pmatrix} 2/3 \\ -1/3 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

11 Linear Combinations and *Span*

Definition. If V is a vector space and $v_1, v_2, \ldots, v_r$ are elements of V (vectors), then a *linear combination of* $v_1, v_2, \ldots, v_r$ is an element of V having the form $c_1v_1 + c_2v_2 + \ldots + c_rv_r$ for some scalars $c_1, c_2, \ldots, c_r$.
Proposition. The set of all linear combinations of $v_1, v_2, \ldots, v_r$ is a linear subspace of V.
The proof is obvious.
Definition. The set of all linear combinations of $v_1, v_2, \ldots, v_r$ is called the *linear span of* $v_1, v_2, \ldots, v_r$ or may also be called the *linear subspace of* V *generated by* $v_1, v_2, \ldots, v_r$.

12 The Row and Column Spaces of a Matrix

Suppose M is an $m \times n$ matrix with
  columns $M_1, M_2, \ldots M_n$
  rows $M^1, M^2, \ldots M^m$ (superscripts)
Definition.
  The *column space* of M is the linear span of $M_1, M_2, \ldots M_n$.

  The *row space* of M is the linear span of $M^1, M^2, \ldots M^m$.
Proposition. If $\mathbf{R}^n \xrightarrow{f_M} \mathbf{R}^m$ is the linear map given by $f_M(x) \;=\; Mx$, then the image of f_M is the column space of M.
Proof. The nature of matrix multiplication is such that
$$Mx \;=\; x_1M_1 + x_2M_2 + \ldots x_nM_n \quad .$$