Linear Algebra (Math 220)
Midterm Test Solutions

March 18, 2008

1. Find the reduced row echelon forms of the following matrices:

$$\text{(a)} \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 2 & -6 & 0 \\ -3 & 9 & 1 \end{pmatrix} .$$

Response.

$$\text{(a)} \quad R_1 \to R_1 - R_2 : \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \qquad R_2 \to R_2 - 3R_1 : \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad R_1 \to R_1 + R_2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{(b)} \quad R_1 \to (1/2)R_1 : \begin{pmatrix} 1 & -3 & 0 \\ -3 & 9 & 1 \end{pmatrix} \qquad R_2 \to R_2 + 3R_1 : \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Let f be the linear function from $\mathbf{R}^4$ to $\mathbf{R}^4$ given by $f(x) = Mx$ where M is the 4×4 matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix} .$$

Find $f(x)$ when x is:

$$\text{(a)} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} .$$

Response. In each case multiply M by x to get:

$$\text{(a)} \begin{pmatrix} 1 \\ -5 \\ -1 \\ 1 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \end{pmatrix}$$

3. Let M be the 3×3 matrix

$$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} .$$

Find the inverse of M.

Response.

$$\begin{pmatrix} 0 & 2 & 4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad R_1 \leftrightarrow R_2 : \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left.\begin{matrix} R_2 \to (1/2)R_2 \\ R_3 \to R_3 - 3R_1 \end{matrix}\right\} : \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & -3 & 0 & -3 & 1 \end{pmatrix} \qquad R_3 \to R_3 - R_2 : \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1/2 & 0 & 0 \\ 0 & 0 & -5 & -1/2 & -3 & 1 \end{pmatrix}$$

$$R_3 \to -(1/5)R_3 : \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/10 & 3/5 & -1/5 \end{pmatrix} \qquad \left.\begin{matrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 2R_3 \end{matrix}\right\} : \begin{pmatrix} 1 & 0 & 0 & -1/10 & 2/5 & 1/5 \\ 0 & 1 & 0 & 3/10 & -6/5 & 2/5 \\ 0 & 0 & 1 & 1/10 & 3/5 & -1/5 \end{pmatrix}$$

Hence,

$$M^{-1} = \begin{pmatrix} -1/10 & 2/5 & 1/5 \\ 3/10 & -6/5 & 2/5 \\ 1/10 & 3/5 & -1/5 \end{pmatrix} .$$

4. Let g be the linear function from $\mathbf{R}^3$ to $\mathbf{R}^3$ that is defined by

$$g(x) = \begin{pmatrix} 1 & 10 & 22 \\ 1 & -2 & -4 \\ 2 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

(a) Find a parametric representation of, or a basis for, the kernel of g.

(b) Find one or more equations in three variables that characterize the image of g.

Response. Manuever a generic augmented matrix so that its first 3 columns are brought to reduced row echelon form:

$$\left.\begin{array}{r} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}\right\} : \begin{pmatrix} 1 & 10 & 22 & y_1 \\ 0 & -12 & -26 & y_2 - y_1 \\ 0 & -18 & -39 & y_3 - 2y_1 \end{pmatrix} \quad R_2 \to -(1/12)R_2 : \begin{pmatrix} 1 & 10 & 22 & y_1 \\ 0 & 1 & 13/6 & (y_1 - y_2)/12 \\ 0 & -18 & -39 & y_3 - 2y_1 \end{pmatrix}$$

$$R_3 \to R_3 + 18R_2 : \begin{pmatrix} 1 & 10 & 22 & y_1 \\ 0 & 1 & 13/6 & (y_1 - y_2)/12 \\ 0 & 0 & 0 & (2y_3 - 3y_2 - y_1)/2 \end{pmatrix} \quad R_1 \to R_1 - 10R_2 : \begin{pmatrix} 1 & 0 & 1/3 & (y_1 + 5y_2)/6 \\ 0 & 1 & 13/6 & (y_1 - y_2)/12 \\ 0 & 0 & 0 & (2y_3 - 3y_2 - y_1)/2 \end{pmatrix}$$

(a) The matrix has rank 2. y_3 may be used as a parameter for its null space (the kernel of g):

$$t \begin{pmatrix} -1/3 \\ -13/6 \\ 1 \end{pmatrix}$$

(b) The image of g is the column space of the matrix. An equation for it is:

$$y_1 = 2y_3 - 3y_2$$

5. Let $\mathcal{P}_2$ be the vector space of all polynomials $at^2 + bt + c$ having degree at most 2 in the variable t. Define $\mathcal{P}_2 \xrightarrow{\phi} \mathcal{P}_2$ by

$$\phi(f(t)) = f''(t) - 3f'(t) + 2f(t) .$$

Find the matrix of ϕ relative to the basis $\mathbf{v}$ of $\mathcal{P}_2$ (playing the role of basis for both the domain and the target of ϕ) given by

$$\mathbf{v} = \{1, t, t^2\} .$$

Response. One computes ϕ at each of the three polynomials in $\mathbf{v}$:

$$\begin{aligned} \phi(1) &= 2 \\ \phi(t) &= -3 + 2t \\ \phi(t^2) &= 2 - 6t + 2t^2 \end{aligned}$$

The coefficient vectors of these ϕ values relative to the basis $\mathbf{v}$ (in its role as basis of the target) are:

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -6 \\ 2 \end{pmatrix} .$$

Hence, the matrix of ϕ with respect to $\mathbf{v}$ is:

$$\begin{pmatrix} 2 & -3 & 2 \\ 0 & 2 & -6 \\ 0 & 0 & 2 \end{pmatrix} .$$