Linear Algebra (Math 220)
Assignment due Tuesday, April 29

Preparation

Expect a **quiz**.

The Characteristic Equation

If, relative to a given coordinate system in an n-dimensional vector space V, the columns of an invertible $n \times n$ matrix Q form a basis of that vector space relative to which a linear transformation that is represented in the given coordinate system by a matrix M is diagonalized, i.e., represented by a diagonal matrix D, then

$$Q^{-1}MQ \;=\; D \quad .$$

Equivalently $MQ = QD$, and, taking the j^{th} column one sees that

$$MQ_j \;=\; (MQ)_j \;=\; (QD)_j \;=\; QD_j \;=\; d_{jj}Q_j \quad .$$

Thus, the member Q_j of the diagonalizing basis must lie in the kernel of the linear function represented in the given coordinate system by the matrix $M - d_{jj}1_n$, where 1_n denotes the $n \times n$ identity matrix. Thus, each Q_j may be found by computing the kernel of $M - t1_n$ when $t = d_{jj}$, and the diagonal elements d_{jj} of D may be found among the roots of the *characteristic polynomial* of M

$$\det\left(M - t1_n\right) \;=\; 0 \quad .$$

A root of the characteristic polynomial is called an *eigenvalue* of M and a coordinate column $v \neq 0$ with the property that $Mv = \lambda v$ for some eigenvalue λ of M is called an *eigenvector* of M. (Moreover, the eigenvalues of M and the elements of V represented in the given coordinate system by the eigenvectors of M may also be called eigenvalues and eigenvectors of the underlying linear transformation of V.)

Exercises

1. Is

$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

   the matrix of the reflection in some line?

2. Find the matrix of the reflection of $\mathbf{R}^3$ in the plane

$$6x - 2y + 3z \;=\; 0$$

3. Find the characteristic polynomial and its roots for each of the matrices

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \quad .$$

4. Let S be the 3×3 matrix

$$\begin{pmatrix} 10 & -6 & -2 \\ -6 & 5 & -8 \\ -2 & -8 & 3 \end{pmatrix} \quad .$$

   Find an orthogonal matrix U and a diagonal matrix D such that

$$S \;=\; UDU^{-1} \quad .$$