Math 220 Quiz Solution

March 6, 2008

The Question

Which sets of column indices (which are integers from 1 to 4) correspond to maximal linearly independent subsets of the set of columns of the matrix

$$M = \begin{pmatrix} 10 & 0 & 1 & -14 \\ -5 & -1 & 0 & 7 \\ 15 & 0 & 0 & -21 \end{pmatrix}$$

Systematic Response

The RREF of M is found to be:

$$\begin{pmatrix} 1 & 0 & 0 & -7/5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$Mx = 0$ occurs when

$$\begin{aligned} x_1 - (7/5)x_4 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 7/5 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{for some } t \quad .$$

Hence,

$$\frac{7}{5}M_1 + M_4 = 0$$

is, up to a scalar multiple, the only linear relation among the columns. It reflects the parallelism of column 1 and column 4. Thus, subsets of the set of column indices corresponding to maximal linearly independent sets of columns are

$$\{1,2,3\} \text{ and } \{2,3,4\} \quad .$$

Remark. The preceding also shows that the vector $(7/5, 0, 0, 1)$ is a basis of the kernel of the linear map $f_M : \mathbf{R}^4 \longrightarrow \mathbf{R}^3$.

An Alternative: "Winging It"

This method avoids all tedious calculation, but it's only a reasonable approach in special circumstances. Here special circumstances are (1) the appearance of vectors on coordinate axes in columns 2 and 3 and (2) the nearly obvious fact that columns 1 and 4 are parallel to each other.

Columns 2 and 3 span the plane in $\mathbf{R}^3$ containing the first two axes, and, therefore, they form a basis of that plane. Neither column 1 nor column 4 is in the plane determined by the first two axes. Hence, the matrix has rank 3. Moreover, $\{1,2,3\}$ and $\{2,3,4\}$ are index sets for maximal linearly independent sets of columns. There are only two other index sets of size 3 — the size required for a maximal linearly independent subset of the set of columns in a rank 3 matrix —, and both of the other sets contain both of the two parallel columns.