Math 220 Class Slides

http://math.albany.edu/pers/hammond/course/mat220/
Course Assignments Slides

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1 Matrix of a Linear Map Relative to a Pair of Bases

The transport diagram:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \alpha_{\mathbf{v}} \uparrow & & \uparrow \alpha_{\mathbf{w}} \\ \mathbf{R}^n & \xrightarrow[f]{} & \mathbf{R}^m \end{array}$$

The linear map f between Euclidean spaces has a matrix M

$$f(x) \;=\; f_M(x) \;=\; Mx$$

Definition. M is called the *matrix of ϕ for the pair of bases*

$$\mathbf{v} \;=\; (v_1 v_2 \ldots v_n) \;\text{ and }\; \mathbf{w} \;=\; (w_1 w_2 \ldots w_m) \quad .$$

2 A Characterization of the Matrix

$$\begin{array}{rl} \text{Basis of } V\text{:} & \mathbf{v} \;=\; (v_1 v_2 \ldots v_n) \\ \text{Basis of } W\text{:} & \mathbf{w} \;=\; (w_1 w_2 \ldots w_m) \\ \text{Standard basis of } \mathbf{R}^n\text{:} & \mathbf{e} \;=\; (e_1 e_2 \ldots e_n) \\ \text{Standard basis of } \mathbf{R}^m\text{:} & \mathbf{e}' \;=\; (e'_1 e'_2 \ldots e'_n) \end{array}$$

$$\begin{array}{lcccl} v_j & & & & \phi(v_j) \;=\; \mathbf{w}M_j \;=\; \sum_i M_{ij} w_i \\ & V & \xrightarrow{\phi} & W & \\ & \alpha_{\mathbf{v}} \uparrow & & \uparrow \alpha_{\mathbf{w}} & \\ & \mathbf{R}^n & \xrightarrow[f_M]{} & \mathbf{R}^m & \\ e_j & & & & Me_j \end{array}$$

$$\boxed{M \text{ is characterized by the formula } \phi(v_i) \;=\; \mathbf{w}M_i}$$

3 Matrix for a $\pi/2$ Rotation in $\mathbf{R}^3$

- **Question.** If P is the plane in $\mathbf{R}^3$ that is the linear span of the vectors

$$v_1 \;=\; \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \;,\quad v_2 \;=\; \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \;,$$

and ρ is the rotation in space through $\pi/2$ about the axis through the origin that is perpendicular to P, specify a basis $\mathbf{v} = (v_1 v_2 v_3)$ of $\mathbf{R}^3$ relative to which the matrix of ρ is relatively simple.

- **Answer.** There is slight ambiguity since it is not possible to distinguish between clockwise and counterclockwise.

  (v_1, v_2) is a basis of the plane P

  One computes the "dot product":

$$v_1 \cdot v_2 \;=\; 1 \cdot 2 + 2 \cdot 1 + (2)(-2) \;=\; 0$$

  So v_1 and v_2 are perpendicular.

  One of the two possible rotations ρ through $\pi/2$ will satisfy:

$$\rho(v_1) \;=\; v_2 \quad \text{and} \quad \rho(v_2) \;=\; -v_1 \quad .$$

  The "cross product" $v_1 \times v_2$ lies on the axis of rotation:

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

  Take $v_3 \;=\; \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, a vector on the axis, as a third basis vector for $\mathbf{R}^3$.

$$\rho(v_3) \;=\; v_3$$

  With $\mathbf{v} \;=\; (v_1 v_2 v_3) \;=\; \mathbf{w}$ as selected pairs of bases, the matrix of ρ is:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

4 Standard Matrix for the $\pi/2$ Rotation

- We have:

$$\mathbf{v} \;=\; (v_1 v_2 v_3) \;=\; \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \;=\; Q$$

$$\rho(\mathbf{v}) \;=\; (\rho(v_1)\rho(v_2)\rho(v_3)) \;=\; (v_1 v_2 v_3) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;=\; \mathbf{v}K$$

- **Note:** The second 3×3 matrix K is the matrix of the linear map ρ with respect to the basis pair $\mathbf{w} \;=\; \mathbf{v}$.

  The first 3×3 matrix Q is not the matrix of a linear map but, rather, a matrix whose columns are the standard coordinates — coordinates with respect to the standard basis — of the members of the basis $\mathbf{v}$.

  The matrix corresponding in a similar way to the standard basis $\mathbf{e} \;=\; (e_1 e_2 e_3)$ is the identity matrix, and it would be more precise, instead of writing $\mathbf{v} \;=\; Q$ to use Q to relate the row of vectors $\mathbf{v}$ to the row of vectors $\mathbf{e}$:

$$\mathbf{v} \;=\; \mathbf{e}Q \quad .$$

  Q is the *matrix for change of basis* between the basis $\mathbf{v}$ and the standard basis $\mathbf{e}$.

- For the standard matrix M of ρ one has $\rho(e_j) = Me_j$ or

$$\rho(\mathbf{e}) = (\rho(e_1)\rho(e_2)\rho(e_3)) = (e_1e_2e_3)M = \mathbf{e}M \quad .$$

- Since ρ is linear, and $\mathbf{v} = \mathbf{e}Q$, one has

$$\rho(\mathbf{v}) = \rho(\mathbf{e}Q) = \rho(\mathbf{e})Q\,,$$

  and, therefore, combining the various formulas:

$$\mathbf{e}QK = \mathbf{v}K = \rho(\mathbf{v}) = \rho(\mathbf{e}Q) = \rho(\mathbf{e})Q = \mathbf{e}MQ$$

  yielding the following relation among ordinary 3×3 matrices:

$$QK = MQ \quad \text{or} \quad M = QKQ^{-1}$$

- Because this particular matrix Q consists of mutually perpendicular columns, all of the same length, it is particularly easy to invert:

$$Q^{-1} = (1/9)\,{}^tQ = (1/9)Q = (1/9)\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \quad .$$

- $$M = \frac{1}{9}\begin{pmatrix} 4 & -1 & 8 \\ -7 & 4 & 4 \\ -4 & -8 & 1 \end{pmatrix} \quad .$$

- M is the standard matrix of one of the two rotations through the angle $\pi/2$ about the line through the origin and the point $(2, -2, 1)$.