Math 220 Class Slides

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1 Matrix Arithmetic

Matrices of the same size can be:

- Added elementwise
- Multiplied by a scalar

For matrices
A of size $m \times n$
B of size $n \times p$
the matrix product
$C = AB$ has size $m \times p$
C (the product) is defined by

$$C_{(i,j)} = \text{(i-th row of A)} \cdot \text{(j-th column of B)}$$

2 Matrix Arithmetic & Linear Equations

A system of linear equations with

- coefficient matrix M of size $m \times n$
- "right-hand side" Y with n coordinates

is essentially the same thing as the matrix equation

$$MX \;=\; Y$$

where

$$X \;=\; \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Note the sizes:

$$M : m \times n \qquad X : n \times 1 \qquad Y : m \times 1$$

3 Matrix Arithmetic & Linear Maps

An $m \times n$ matrix determines a "linear map" f_M:

$$Y \;=\; f_M(X) \;=\; MX$$

- M has size $m \times n$
- X is a column of length n — size $n \times 1$
- MX is a column of length m — size $m \times 1$
- f_M sends a point X in n-dimensional space $\mathbf{R}^n$ to a point Y of m-dimensional space $\mathbf{R}^m$. Notation:

$$\mathbf{R}^n \xrightarrow{f_M} \mathbf{R}^m$$

4 Exercises 1 & 3

The linear system

$$MX = Y$$

with

$$M = \begin{pmatrix} 1 & -1 & 1 \\ 5 & -4 & 3 \\ 3 & -3 & 2 \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad Y = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

has solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -u+v-w \\ u+v-2w \\ 3u-w \end{pmatrix}$$

or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & -2 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = Q \begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad Q = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & -2 \\ 3 & 0 & -1 \end{pmatrix}$$

5 Exercises 1 & 3 Retrospective

- The execution of row operations shows that

$$X = QY \quad \text{if} \quad MX = Y$$

- Each elementary row operation can be reversed by another.
-

$$MX = Y \quad \text{if and only if} \quad X = QY$$

- Morever,

$$(MQ)Y = M(QY) = MX = Y \text{ for every } Y = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

- So

$$f_{MQ} = f_M \circ f_Q = \text{the identity map} \quad \text{and} \quad MQ = 1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6 About invertible matrices

A *square* matrix is invertible if there are no zero rows in its row echelon form(s).

7 Exercise No. 4

A slightly different matrix

$$N = \begin{pmatrix} 1 & -2 & 1 \\ 5 & -4 & 3 \\ 3 & -3 & 2 \end{pmatrix}$$

- A square matrix
- Its row echelon forms have two non-zero rows and one zero row
- Row reduction for the linear system

  $$NX = Y = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad \text{(arbitrary right-hand side, as above)}$$

  leads to (in the last row)

  $$0 = u + v - 2w$$

- Only one of the three given right-hand sides admits solutions
- $$NX = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{if and only if} \quad X = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \text{ any } t$$