Linear Algebra (Math 220)
Assignment due Thursday, May 1

Diagonalization and Orthogonal Diagonalization

Relevant reading: Lay § 7.1

- Two $n \times n$ matrices A, B are called *similar* when there is an invertible matrix Q such that $Q^{-1}AQ = B$.
- An $n \times n$ matrix M is called *diagonalizable* when it is similar to a diagonal matrix.
- If M is an $n \times n$ matrix that is the matrix of a linear map $V \xrightarrow{\varphi} V$ relative to a basis of V, then M is diagonalizable if and only if there is some basis of V consisting of eigenvectors of M.
- Two $n \times n$ matrices A, B are called *orthogonally similar* when there is an orthogonal matrix U such that $U^{-1}AU = B$.
- An $n \times n$ matrix M is called *orthogonally diagonalizable* when it is orthogonally similar to a diagonal matrix.
- If V is a vector space with a given inner product I, a linear map $V \xrightarrow{\varphi} V$ is called *symmetric* relative to I if and only if for all choices of v_1, v_2 in V one has $I(\varphi(v_1), v_2) = I(v_1, \varphi(v_2))$.
- If M is an $n \times n$ matrix that is the matrix of a linear map $V \xrightarrow{\varphi} V$ with respect to a basis that is orthonormal relative to an inner product I, then the following conditions are equivalent:
  1. M is a symmetric matrix.
  2. φ is symmetric relative to I.
  3. M is orthogonally diagonalizable.
  4. There is some orthonormal basis of V consisting of eigenvectors of φ.

Exercises

1. Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of the matrix
$$\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix} .$$
2. Give an example of a 2×2 matrix having eigevalues 1 and -1 where the corresponding eigenvectors form the angle $\pi/4$.
3. Show that the matrix
$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$
is not similar to a diagonal matrix.
4. Let S be the 3×3 symmetric matrix
$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} .$$
   (a) Find an orthogonal matrix U and a diagonal matrix D such that
$$U^{-1}SU = D \quad .$$
   (b) What is the largest value achieved on the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ by the function
$$h(x) = {}^t xSx = 2x_1^2 + 3x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 \text{ ?}$$
5. What geometric property might be said to characterize the $n \times n$ matrices that are similar to upper triangular matrices?