Algebra Preliminary Exam

January 1997

1. Prove that if A is an $n \times n$ matrix with coefficients in a field, then A is similar to a matrix of the form $\begin{pmatrix} A_1 & & & & 0 \\ & A_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & A_r \end{pmatrix}$ where the characteristic polynomial of A_i $(i = 1 \ \ldots \ r)$ is the power of an irreducible polynomial.

2. Prove that $(p-1)! \equiv -1 \pmod{p}$ for p an odd prime.

3. If G is any group and H is a subgroup of G with $G : H = n$, show that there exists a normal subgroup K of G such that $K \subseteq H$ and $G : K \leq n!$

4. Determine the structure of the Galois group G of the splitting field M over the rational numbers Q of the polynomial $f(x) = x^5 - 2$. How many Sylow 2-subgroups does G have? Give the fixed subfields of M of each Sylow 2-subgroup. Do the same thing for the Sylow 5-subgroups. Which of these subfields are normal field extensions of Q?

5. Let K be a normal, separable extension field of F, and $p(x) \in F[x]$ be an irreducible polynomial. If in $K[x]$ $p(x) = p_1(x) \cdot \ldots \cdot p_r(x)$ where $p_i(x)$ are irreducible polynomials in $K[x]$, $i = 1 \ldots r$, prove that $p_1(x), \ldots, p_r(x)$ all have the same degree.

6. Let R be an integral domain. State and prove the universal mapping property for the embedding of R into its field of fractions.

7. Let R be a ring with unit, A, C right R-modules, B, D left R-modules, $f : A \to C$ a right R-module homomorphism, $g : B \to D$ a left R-module homomorphism. Let $h : A \otimes_R B \to C \otimes_R D$ be defined by $h(a \otimes b) = f(a) \otimes g(b)$. If f and g are monomorphisms, is h necessarily a monomorphism? Why?

8. Let p be a prime number and $\mathbf{Z}_p$ be the completion of $\mathbf{Z}$ at the prime ideal $p\mathbf{Z}$. Prove that there exists a map

$$\chi : \mathbf{F}_p \to \mathbf{Z}_p$$

with the following properties:

(a) If $\pi : \mathbf{Z}_p \to \mathbf{F}_p$ is the canonical map, then

$$\pi \cdot \chi \text{ is the identity on } \mathbf{F}_p$$

(b) χ is multiplicative: that is, $\chi(ab) = \chi(a)\ \chi(b)$ for all a, b in $\mathbf{F}_p$.