Ph.D. Preliminary Examination

ALGEBRA

Fall 1995

1. Let K be a field, L a field extension of K. An element α in L is algebraic over K if α is the root of some monic polynomial with coefficients in K.

   Show that if α and β in L are algebraic over K, then $\alpha\beta$ is algebraic over K.

2. Let w be a primitive cube root of unity. Let $R = \mathbf{Z}[w]$. Let $\lambda = 1 - w$. Show that $R/\lambda R \cong \mathbf{Z}/3\mathbf{Z}$.

3. Let G be the group of 2×2 invertible matrices of determinant 1 with coefficients in the field of 3 elements.

   (a) Show that G has order 24.

   (b) Find the number of 3-Sylow subgroups of G.

4. Let G be a finite p-group, p prime, V a finite dimensional vector space over the field $\mathbf{F}_p$ of p elements. Suppose G acts linearly on V (i.e. there is a homomorphism from G into the group $GL(V)$ of invertible linear transformations from V to V). Prove that G has a non-zero fixed point: that is, there is some $\alpha \neq 0$ in V so that $\sigma(\alpha) = \alpha$ for all σ in G.

5. Let L/K be a Galois extension of fields with Galois group G. Let $L = K[\alpha]$. Define $tr(\alpha) = \sum_{r \in G} \sigma(\alpha)$. Let $T_\alpha : L \to L$ be the K-linear transformation defined by $T_\alpha(\beta) = \alpha\beta$. Show that $tr(\alpha)$ is the trace of the linear transformation T_α.

6. Prove that for any prime p, there are at least four isomorphism classes of groups of order p^3.

7. A $\mathbf{Z}$-module M is flat if for any short exact sequence $0 \to A \to B \to C \to 0$ of $\mathbf{Z}$-modules, the sequence $0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0$ is exact.

   (a) State and prove a criterion for flatness as follows: M is flat if and only if for any homomorphism $f : E \to F$ of $\mathbf{Z}$-modules, if t is ____jective, then $M \otimes f$ is ____jective.

   (b) Give an example of a non-flat $\mathbf{Z}$-module.

8. Let K be a field, M a K-vector space. Let $M^* = \mathrm{Hom}_R(M, K)$. Show that the canonical map $M \to M^{**}$ is surjective if and only if M is finite dimensional.