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Preliminary Ph.D. Examination in Algebra

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[10] 1. Let $\mathbf{Q}$ denote the field of rational numbers. Prove **or** disprove **one** of the following assertions concerning an arbitrary *symmetric* matrix M with entries in $\mathbf{Q}$. There exists an invertible matrix P with entries in $\mathbf{Q}$ such that:

(a) $PM\,{}^tP$ is diagonal.

(b) PMP^{-1} is diagonal.

[10] 2. Let k be a field.

(a) Show that $(x+1)$ is a maximal ideal in the polynomial ring $k[x]$.

(b) Show that $(x+1,\ y-2)$ is a maximal ideal of $k[x,y]$.

(c) Let A be the quotient ring $k[x,y]/(x+1)$, and let φ be the $k[x,y]$-linear endomorphism of A given by

$$\varphi(f) = (y-2)f\ .$$

Show that the cokernel of φ is a 1-dimensional k-module.

[10] 3. Show that every automorphism of the symmetric group S_3 (the group of all permutations of a set with 3 members) is an inner automorphism.

[10] 4. Let E be a (finite) Galois extension field of F with Galois group G; let K be an intermediate field and H the subgroup of G that fixes each member of K. Show that the subgroup of G consisting of all σ in G for which $\sigma(K) = K$ is the *normalizer* of H in G, i.e., the largest subgroup N of G containing H for which H is a normal subgroup of N.

[10] 5. For K any field $\mathrm{GL}(n,K)$ denotes the group of invertible $n \times n$ matrices in the field K, and $\mathrm{SL}(n,K)$ denotes the group of such matrices of determinant 1. Prove that $\mathrm{GL}(n,K)$ is isomorphic to a semi-direct product of $\mathrm{SL}(n,K)$ with $\mathrm{GL}(1,K)$.

[10] 6. Let R be a commutative ring with a **unique** maximal ideal M. Show that if $e^2 = e$, then $e = 0$ or $e = 1$.

[10] 7. For I and J ideals in a commutative ring R, prove that the natural R-algebra homomorphism

$$R/I \otimes_R R/J \longrightarrow R/(I+J)$$

is an isomorphism of R-algebras.

[10] 8. Let K be the splitting field over the field $\mathbf{Q}$ of rational numbers of the polynomial

$$t^4 + 4t^2 + 2\ .$$

Find the Galois group of K over $\mathbf{Q}$.