Algebra Preliminary Exam

January 20, 1995

1. Prove or disprove the following assertion:

   Every real symmetric matrix has a unique real symmetric cube root.

2. Let $\mathbf{Z}[x]$ be the ring of polynomials in one variable with coefficients in the ring $\mathbf{Z}$ of integers. Let I be the ideal of all polynomials $f(x)$ in $\mathbf{Z}[x]$ such that $f(0) = 0$, and let J be the ideal of all polynomials $f(x)$ in $\mathbf{Z}[x]$ such that $f(0)$ is an even integer. Show that:

   a) I is a prime ideal.

   b) J is a maximal ideal.

   c) I is a principal ideal.

   d) J is **not** a principal ideal.

3. Let E be the splitting field over the field F of the polynomial $t^{15} - 1$. Determine the extension degree $[E : F]$ when F is:

   (a) the field $\mathbf{R}$ of real numbers.

   (b) the field $\mathbf{F}_2$ of integers mod 2.

   (c) the field $\mathbf{F}_{31}$ of integers mod 31.

   (d) the field $\mathbf{Q}$ of rational numbers. **Hint**: The splitting fields of $t^5 - 1$ and of $t^3 - 1$ are subfields. Show that the intersection of these two subfields is $\mathbf{Q}$.

4. Let E be a (finite) Galois extension field of F with Galois group G; let K be an intermediate field and H the subgroup of G that fixes K. Show that the subgroup of G consisting of all σ in G for which $\sigma(K) = K$ is the normalizer of H in G.

5. Prove directly, without quoting the structure theorem for finitely generated modules, that in a principal ideal domain the matrices $\begin{bmatrix} a & 0 \\ 0 & b) \end{bmatrix}$ and $\begin{bmatrix} gcd(a,b) & 0 \\ 0 & lcm(a,b) \end{bmatrix}$ are row-and-column equivalent.

6. Suppose that for a certain integer $n > 1$, every group of order n is cyclic. Prove that n is relatively prime to $\phi(n)$.

7. Let ℓ denote "length" (in the sense of the Jordan-Holder theorem). Complete the following statement concerning the nontrivial abelian groups A and B ,and then prove the assertion:

$$\ell(A) = \ell(B) \text{if and only if } \ldots \ .$$

8. Let R be a commutative ring. Let M be a R-module. Consider the functors $- \otimes M$, $\mathrm{Hom}(M, -)$, $\mathrm{Hom}(-, M)$ on the category of R-modules.
   (a) List the exactness properties of each of these three functors in general and prove in detail what you have said about one of them.
   (b) What more can you say if M is R-free?