Department of Mathematics and Statistics

Preliminary Ph.D. Examination in Algebra

January 18, 1994

1. (a) Let G be a non-abelian group of order 8. Prove that there exists a cyclic subgroup H of order 4 in G. Determine whether this subgroup H is normal in G.

   (b) Give an example of a group of order 8 which does not admit any cyclic subgroup of order 4.

2. Let K be a finite field of cardinality q.

   (a) Show that K has characterstic p for some prime p.

   (b) Show that the group of ring automorphisms $\mathrm{Aut}(K)$ is cyclic of order $\log_p q$.

3. Let A_4 denote the group of "even" permutations of a set of cardinality 4.

   (a) Determine the 2-Sylow subgroup(s) of A_4.

   (b) Let T be a 3-Sylow subgroup of A_4. Show that T is not normal in A_4.

   (c) Show that any group of order 12 which does not have a normal 3-Sylow subgroup is isomorhic to A_4.

4. Let $\mathbf{Q}$ denote the field of rational numbers. Let K denote the field obtained by adjoining all the complex third roots of 2 to $\mathbf{Q}$.

   (a) Determine the degree $[K : \mathbf{Q}]$.

   (b) Determine the Galois group of the extension $K/\mathbf{Q}$.

   (c) Determine all the subfields of K.

5. Let A be a ring and $0 \to K \to F \xrightarrow{\pi} M \to 0$, $0 \to K' \to F' \xrightarrow{\pi'} M \to 0$ be two short exact sequences of left A-modules, where F and F' are free A-modules. Show that $F \oplus K' \cong F' \oplus K$ as left A-modules as follows:

   Let $Y = \{(f, f') \in F \oplus F' | \pi(f) = \pi'(f')\}$.

   Let $g : Y \to F$ by $g(f, f') = f$ and let $g' : Y \to F'$ by $g'(f, f') = f'$.

   Show that

   (i) g and g' are surjective.

   (ii) $Y \cong F \oplus \ker(g) \cong F' \oplus \ker(g')$.

   (iii) $\ker(g) \cong K'$ and $\ker(g') \cong K$.

   Hence $F \oplus K' \cong F' \oplus K$.

6. Let R be a commutative ring.

   (a) Show that P is a projective R-module iff $\mathrm{Hom}(P, -)$ when applied to a short exact sequence of R-modules is exact.

   (b) Prove that every free module is projective.

   (c) Give an example of a projective which is not free.

   (d) Classify all projective modules over principal ideal domains.

7. (a) Show that every commutative ring with identity has a maximal ideal.

   (b) Give an example of a ring with a unique maximal nonzero ideal.

   (c) Give an example of a ring with a finite (> 1) number of maximal ideals.

   (d) Give an example of a ring with an infinite number of maximal ideals.

8. Let $M_n(F)$ denote the ring of n-by-n matrices with entries in a field F.

   Let $E(i, j)$ denote the matrix with (i, j)-entry equal to 1 and all other entries equal to 0. Let A be a matrix in $M_n(F)$.

   (a) Describe the result of multiplication of A by $E(i, j)$ on the left and on the right respectively.

   (b) Show that $M_n(F)$ has no non-trivial proper two-sided ideals.