DEPARTMENT OF MATHEMATICS & STATISTICS

Preliminary Ph.D. Examination in Algebra

September 2, 1993

1. Determine the number of 3×3 invertible matrices in a finite field having q elements.

2. Can a line segment with length equal to the positive real fifth root of 2 be constructed (given unit length) in a finite number of steps using straightedge and compass? Explain.

3. Let A be a commutative ring (with multiplicative identity).

   (a) Let M be an A-module. Let $R = \text{End}(M)$ be the set of endomorphisms of M (i.e., the set of A-module homomorphisms $M \to M$). Define operations on R that make R an A-algebra (i.e., a ring with compatible A-module structure).

   (b) Is the set of ring endomorphisms (as opposed to A-module endomorphisms) of the ring A a ring?

   (c) When $A = \mathbf{Z}$, the ring of integers, find the endomorphism ring of the $\mathbf{Z}$-module $\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$.

4. Show that any group of order 20 has a non-trivial proper normal subgroup.

5. Prove that a finitely-generated torsion-free module over a principal ideal domain is necessarily free.

6. Determine the isomorphism class of each of the Sylow subgroups of the alternating group A_5, the group of "even" permutations of a set of cardinality 5.

7. Let ζ be a primitive 7^{th} root of unity in the field of complex numbers, let $K = \mathbf{Q}(\zeta)$, and $H = \mathbf{Q}(\alpha)$, where $\alpha = cos(2\pi/7)$ and $\mathbf{Q}$ denotes the field of rational numbers.

   (a) Show that $H = K \cap \mathbf{R}$, where $\mathbf{R}$ is the field of real numbers.

   (b) Prove that K and H are both normal extensions of $\mathbf{Q}$.

   (c) Determine the Galois groups $\text{Gal}(K : \mathbf{Q})$, $\text{Gal}(H : \mathbf{Q})$, and $\text{Gal}(K : H)$.

8. For p a prime the ring of p-adic integers $\mathbf{Z}_p$ is defined to be the inverse limit of the unique ring homomorphisms

$$\ldots \to \mathbf{Z}/p^n\mathbf{Z} \to \ldots \to \mathbf{Z}/p^2\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \,.$$

   Let π denote the canonical ring homomorphism $\mathbf{Z}_p \to \mathbf{Z}/p\mathbf{Z}$.

   (a) Show that an element of $\mathbf{Z}_p$ is invertible in $\mathbf{Z}_p$ if and only if its image under π is non-zero.

   (b) Show that the kernel of π is a maximal ideal of $\mathbf{Z}_p$.

   (c) Show that any proper ideal in $\mathbf{Z}_p$ is contained in the kernel of π. (Hence, $\ker(\pi)$ is the only maximal ideal.)

   (d) Show that the kernel of π is a principal ideal.

   (e) Give another (non-isomorphic) example of a ring having a unique maximal ideal that is principal.