Preliminary Examination in Algebra
Department of Mathematics and Statistics
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1. Show that in any group of order $2 \cdot 9 \cdot 17$, the 17-Sylow subgroup is normal. (Hint: What is the normalizer of a subgroup of order 3?)

2. Let G be a group of order $4p^n$, where $p > 2$ is prime and $n > 0$. Show that G is not simple. (Hint: Consider the standard action of G on G/P, where P is a p-sylow subgroup.)

3. Let $F = \mathbb{Q}(\zeta_9)$ with $\zeta_9 = e^{\frac{2\pi i}{9}}$.
   a) What is the Galois group of F over $\mathbb{Q}$?
   b) Find all intermediate fields between $\mathbb{Q}$ and F. (Write each in the form $\mathbb{Q}(\alpha)$ for some specific $\alpha \in F$.)
   c) For each intermediate field E above, give the Galois group of E over $\mathbb{Q}$.

4. The prime factorization of $x^7 - 1$ in $\mathbb{Z}_2[x]$ is
$$x^7 - 1 = (x - 1)(x^3 + x^2 + 1)(x^3 + x + 1).$$
   Recall that for any field F, a matrix $A \in \mathrm{Gl}_n(F)$ satisfies $A^7 = I$ (i.e., A has order dividing 7) if and only if the minimal polynomial of A divides $x^7 - 1$.
   a) Give the rational canonical forms of all elements of order exactly 7 in $\mathrm{Gl}_3(\mathbb{Z}_2)$. (Write down the specific 3×3 matrices.)
   b) Give the rational canonical forms of all elements of order exactly 7 in $\mathrm{Gl}_6(\mathbb{Z}_2)$. Here you may use the $C(f)$ notation and specify appropriate block sums for appropriate f's.