Algebra Preliminary Exam

January 20, 2009

1. Show that every group of order 24 has a nontrivial normal subgroup.

2. Let $p \geq 2$ be a prime. Show that a finite p-group has a nontrivial center.

3. Let p be an odd prime. Show that every nonabelian group of order $2p$ has a trivial center.

4. The eigenvalues of the matrix

$$A = \begin{pmatrix} 3 & 4 & 0 \\ -1 & -3 & -2 \\ 1 & 2 & 1 \end{pmatrix}.$$

over the field of rationals $\mathbb{Q}$ are 1 and -1.

(a) For each eigenvalue compute the dimension over $\mathbb{Q}$ of the corresponding eigenspace.

(b) Find the minimal polynomial of A.

(c) Find the sequence of (polynomial) invariant factors for A, and the rational canonical form of A over $\mathbb{Q}$.

5. Let K be an algebraic field extension of a field F, and let L be a subfield of K such that $F \subseteq L$ and L is normal (but not necessarily finite) over F. Show that if σ is any automorphism of K over F, then $\sigma(L) = L$.

6. (a) Show that the polynomial $x^5 - 5$ is an irreducible polynomial over $\mathbb{Q}$, the field of rational numbers.

(b) Let the complex number ω be a primitive fifth root of unity over $\mathbb{Q}$ and let $f(x) = x^4 + x^3 + x^2 + x + 1$ be the minimal polynomial of ω over $\mathbb{Q}$. Show that $f(x)$ is also the minimal polynomial of ω over $\mathbb{Q}(\sqrt[5]{5})$, and that $x^5 - 5$ is the minimal polynomial of $\sqrt[5]{5}$ over $\mathbb{Q}(\omega)$.

(c) Prove that $\mathbb{Q}(\omega, \sqrt[5]{5})$ is the splitting field of $x^5 - 5$ over $\mathbb{Q}$, and find the Galois group of $x^5 - 5$ over $\mathbb{Q}$.