The University at Albany
Department of Mathematics and Statistics
Ph. D. Program
Preliminary Examination in Algebra
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Problem 1.

a. State the Jordan–Holder Theorem.

b. Give the definition of a solvable group.

c. Give an example of a non–solvable group.

Problem 2. Let p, q be two odd prime numbers with $p < q$. Let $n = pq$. Show that every group of order n is cyclic if and only if p does not divide $q - 1$.

Problem 3. Let S_3 denote the symmetric group of order 6.

a. Determine the group of inner automorphisms of S_3.

b. Show that every automorphism of S_3 is inner.

Problem 4. Prove that there are at least 4 isomorphism classes of groups of order 8.

Problem 5. Find all real 3×3 matrices A, up to similarity, with $A^3 = I$.

Problem 6.

a. Let R be a ring. Define a projective R–module.

b. In case $R = \mathbb{F}$ is a field, give an example of a finitely generated R–module that is not projective or show that such an example does not exist.

c. In case $R = \mathbb{Z}$, the ring of integers, give an example of a finitely generated R–module that is not projective or show that such an example does not exist.

Problem 7. Let $\mathbb{K}$ be a field. Let $\mathbb{K}^x$ denote the multiplicative group of units of $\mathbb{K}$.

a. Show that a finite subgroup G of $\mathbb{K}^x$ is cyclic.

b. Evaluate the following sum in $\mathbb{K}$:

$$\sum_{g \in G} g$$

Problem 8. Let $\mathbb{Q}$ denote the field of rational numbers. Let $\mathbb{K}$ denote the field obtained by adjoining all the complex third roots of 2 to $\mathbb{Q}$.

a. Determine the Galois group of the extension $\mathbb{K}$ over $\mathbb{Q}$.

b. Can a line segment with length equal to $\sqrt[3]{2}$ be constructed (given unit length) in a finite number of steps using straight edge and compass? Explain.