Preliminary Examination in Algebra

Department of Mathematics and Statistics

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1. Show that any group of order 33 is cyclic.

2. Show that any group of order 132 has a normal 11-Sylow subgroup. (Hint: What is the order of the normalizer of a 3-Sylow subgroup?)

3. Let $A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$.
   a) What is the characteristic polynomial of A?
   b) What is the mimimal polynomial of A?
   c) What is the rational canonical form of A?
   d) What is the Jordan canonical form of A?

4. Let $A \in \mathbb{Q}[x]$ with characteristic polynomial f^n, where $f = x^2 + 1$. What is the largest value of n such that the rational canonical form of A is determined by $\mathrm{ch}_A(x)$ together with the dimensions of the nullspaces of $f(A)$ and $f^2(A)$?
   a) Why?
   b) For this n, display a pair of matrices A, B with
   $$\mathrm{ch}_A(x) = \mathrm{ch}_B(x) = f^{n+1}$$
   $$\dim N(f(A)) = \dim N(f(B))$$
   $$\dim N(f^2(A)) = \dim N(f^2(B)),$$
   but whose rational canonical forms are different.

5. Using the analogue for abelian groups of rational canonical form, classify the abelian groups of order 48.

6. What is the Galois group of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$? Here, $\zeta_n = e^{2\pi i/n}$ is the standard primitive n-th root of unity in $\mathbb{C}$.

7. Using $\mathrm{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})$, find all the subfields of $\mathbb{Q}(\zeta_{12})$. (Note: $\zeta_{12} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.)