Preliminary Examination in Algebra

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Directions: There are 8 questions, all of the same weight. Please take the time to ensure accuracy and completeness, especially for the questions you find easiest. (Completeness does not mean excessive verbosity. You should not attempt to prove standard propositions that you cite except when the proof of a standard proposition is explicitly sought.)

The ring of integers will be denoted by $\mathbf{Z}$ and its field of fractions by $\mathbf{Q}$. Unless stipulated otherwise, a ring is always assumed to have an identity element for multiplication, and a module over a ring is always assumed to be unitary, i.e., the multiplicative identity of the ring is assumed to induce via scalar multiplication the identity endomorphism of the abelian group subordinate to the module.

1. Prove that if p, q are primes with $q < p$ and q not a divisor of $p - 1$, then every group of order pq is abelian.

2. Show that if R is an integral domain, then R may be identified as a subring of a field.

3. Let $\alpha = \sqrt{2} + \sqrt[3]{3} + \sqrt[4]{5}$, and let $\mathbf{Q}[\alpha]$ be the set of real numbers obtained by evaluating polynomials with rational coefficients at α. Explain why $\mathbf{Q}[\alpha]$ is a field.

4. Let A be a finite abelian group, written additively. Let λ be the smallest positive integer such that $\lambda a = 0$ for all a in A. Show that there exists an element b of A whose order is λ.

5. Show that the group $G = \mathrm{SL}_2(\mathbf{Z}/3\mathbf{Z})$ has a normal subgroup of order 8, and identify its isomorphism class as one of the 5 isomorphism classes of groups of order 8. *Hint:* G has no element of order 8.

6. Show that $\mathbf{Q}[x_1, x_2, x_3, x_4]/(x_1x_3 - x_2x_4)$ is not a unique factorization domain.

7. Let S_n be the group of permutations of the set $X = \{a_1, \ldots, a_n\}$. A subgroup G of S_n is called *transitive* if the orbit of a_1, under the action of G is all of X.

   Let $f(x)$ be an irreducible polynomial of prime degree p with coefficients in a field K of characteristic zero. Show that the Galois group of $f(x)$ is a transitive subgroup of S_p.

8. Classify (up to isomorphism) all finitely-generated $\mathbf{Q}[x]$-modules having dimension 4 as vector spaces over $\mathbf{Q}$ that are annihilated by the polynomial $x(x + 2)^2$.