Preliminary Examination in Algebra

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Directions: There are 8 questions, all of the same weight. Please take the time to ensure accuracy and completeness, especially for the questions you find easiest. (Completeness does not mean excessive verbosity. You should not attempt to prove standard propositions that you cite except where the proof of a standard proposition is explicitly sought.)

The ring of integers will be denoted by $\mathbf{Z}$ and its field of fractions by $\mathbf{Q}$. Unless stipulated otherwise, a ring is always assumed to have an identity element for multiplication, and a module over a ring is always assumed to be unitary, i.e., the multiplicative identity of the ring is assumed to induce via scalar multiplication the identity endomorphism of the abelian group subordinate to the module.

1. When S is a ring, let S^{op} denote the "opposite" ring, i.e., the "same" additive group with multiplication reversed, and let $\mathrm{Mat}_n(S)$ denote the ring of $n \times n$ matrices in S. Show that for any ring R there is an isomorphism
$$\mathrm{Mat}_n\left(R^{\mathrm{op}}\right) \cong \left(\mathrm{Mat}_n(R)\right)^{\mathrm{op}} \quad .$$

2. Prove that every group of order $4p$, with $p > 5$ a prime, is isomorphic to a semi-direct product NH of a group N of order p with a group H of order 4.

3. Prove that the group, under multiplication, of all non-zero elements in a finite field must be a cyclic group.

4. Let $\mathbf{Z}/m\mathbf{Z}$ denote the ring of integers modulo m. Let r, s be positive integers.
   (a) What element of $\mathbf{Z}$ generates the ideal $r\mathbf{Z} + s\mathbf{Z}$?
   (b) What is the kernel of the canonical ring homomorphism
   $$\mathbf{Z}/rs\mathbf{Z} \longrightarrow \mathbf{Z}/r\mathbf{Z} \times \mathbf{Z}/s\mathbf{Z} \ ?$$
   (c) Use the universal mapping property characterizing the tensor product of $\mathbf{Z}$-algebras to explain why
   $$\mathbf{Z}/r\mathbf{Z} \otimes \mathbf{Z}/s\mathbf{Z} \cong \mathbf{Z}/\left(r\mathbf{Z} + s\mathbf{Z}\right) \quad .$$

5. Classify (up to isomorphism) all finitely-generated $\mathbf{Q}[x]$-modules having dimension 4 as vector spaces over $\mathbf{Q}$ that are annihilated by the polynomial $x(x+2)^2$.

6. Let G be a simple group of order $3420 = 2^2 \cdot 3^2 \cdot 5 \cdot 19$. (The projective special linear group formed from 2×2 matrices over the field $\mathbf{F}_{19}$ is an example.) Show that G has no element of order $171 = 3^2 \cdot 19$. *Hints:* The only divisors of $180 = 3420/19$ congruent to 1 mod 19 are 1 and 20. Consider the action of G by conjugation on the set of Sylow p-subgroups of G for some prime divisor p of 3420.

7. (a) Find $\alpha \in \mathbf{C}$ such that $N = \mathbf{Q}(\alpha, \sqrt[3]{2})$ is the normal closure of $\mathbf{Q}(\sqrt[3]{2})$ over $\mathbf{Q}$.
   (b) Explain why N is a Galois extension of $\mathbf{Q}$.
   (c) Must there exist $\beta \in N$ such that $N = \mathbf{Q}(\beta)$?
   (d) What is the Galois group $\mathrm{Gal}(N/\mathbf{Q})$?
   (e) What are the intermediate subfields of N as an extension of $\mathbf{Q}$, i.e., the fields F for which $N \supset F \supset \mathbf{Q}$.

8. Show that $\mathbf{Q}[x_1, x_2, x_3, x_4]/(x_1x_3 - x_2x_4)$ is not a UFD.