Ph.D. Preliminary Examination in Algebra

August 31, 2006

1. Show that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$, if and only if m and n are relatively prime.

2. Let p be a prime number. How many Sylow p-subgroups does S_p have?

3. Show that there is no simple group of order 160.

4. Show that $\mathbb{Z}[\sqrt{3}]$ is a UFD.

5. Let K be a finite field.

   (a) Show that there exists a prime number p so that K contains a subfield F isomorphic to the field $\mathbb{F}_p$ of p elements.

   (b) Show that there exists a polynomial $q(x)$ with coefficients in F such that K is isomorphic (as rings) to the ring $F[x]/(q(x))$.

   (c) Show that $K : F$ is Galois.

6. (a) Describe the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ and its action on $\mathbb{Q}(\zeta_5)$, where $\zeta_5 = e^{2\pi i/5}$.

   (b) Determine the minimal polynomial of $cos(2\pi/5)$ and show that $cos(2\pi/5) = \frac{-1+\sqrt{5}}{4}$.

   (c) Find the tower of subfields of $\mathbb{Q}(\zeta_5)$ and express them as fixed subfields of subgroups of $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$.

7. Prove that a left module M over a ring with identity R is simple (i.e., $M \neq 0$ and M has no proper submodules) if and only if M is isomorphic to R/I for some maximal left ideal I.

8. If A is an $n \times n$ matrix with entries in a field k, show that A is similar to its transpose A^t.

9. (a) Define projective module.

   (b) Define injective module.

   (c) Prove or disprove: $\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

   (d) Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.