Algebra Preliminary Exam

August, 2003

1) Let $GL_n(F_p)$ denote the group of invertible $n \times n$ matrices with entries in the field of p elements $F_p = Z/pZ$. Equivalently, $GL_n(F_p)$ denotes the group under composition of the 1-1 and onto linear transformations from F_p^n to F_p^n.

   a) Determine the order of $GL_3(F_p)$.

   b) Show that the subgroup U of $GL_3(F_p)$ consisting of matrices of the form $I + N$ where N is upper triangular with zero diagonal, is a p-Sylow subgroup of $GL_n(F_p)$.

   c) Show that U is nilpotent but not abelian.

2) If G is a group, let $Aut(G)$ be the group of automorphisms of G. Define $C : G \to Aut(G)$ to be the function defined by $C(g)(x) = gxg^{-1}$ for g, x in G.

   a) Show that C is a homomorphism.

   b) Describe the kernel of C.

   c) Show that if $G = S_3$, the symmetric group on three symbols, then C is an isomorphism.

3) Define the semi-direct product of two groups as follows: let $(G, *)$ and $(A, \cdot)$ be groups, and $\alpha : A \to Aut(G)$ a homomorphism. Then

$$G \rtimes_\alpha A = \{(g, a) \in G \times A\}$$

with multiplication:

$$(g_1, a_1) \cdot (g_2, a_2) = (g_1 * \alpha(a_1)(g_2), a_1 \cdot a_2).$$

   a) Show that if α is trivial, then $G \rtimes_\alpha A \cong G \times A$.

   b) Find six pairwise non-isomorphic groups of order 42, all of which are semi-direct products.

4) If R is a commutative ring with 1 and I, J are ideals of R so that $I + J = R$, then
$$R/(I \cap J) \cong R/I \oplus R/J.$$
Show that when $R = Z$, the integers, this theorem is equivalent to a well-known theorem in elementary number theory. Explain.

5) Find the Galois group over F_2, the field of 2 elements, of the irreducible polynomial $f(x) = x^5 + x^2 + 1$.

6) How many similarity classes of matrices with complex entries have characteristic polynomial $(x-1)(x-2)^2(x-3)^3$? Explain.

7) Let F be a finite field.

   a) Define the characteristic of F.

   b) Prove that the characteristic of F must be a prime p.

   c) Prove that the cardinality of F must be a power of p.

   d) Show that F is the splitting field of some irreducible polynomial.

8) Suppose R is a ring.

   a) What does it mean to say that R is a *Euclidean Ring*?

   c) What does it mean to say that R is a *Principle Ideal Domain*?

   c) Prove that every Euclidean Ring is a Principle Ideal Domain.