Preliminary Examination in Algebra

Department of Mathematics & Statistics

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Directions: There are 8 questions, all of the same weight. Please take the time to ensure accuracy and completeness, especially for the questions you find easiest. (Completeness does not mean excessive verbosity. You should not attempt to prove standard propositions that you cite except where the proof of a standard proposition is explicitly sought.)

The ring of integers will be denoted by $\mathbb{Z}$ and its field of fractions by $\mathbb{Q}$.

1. Show that every automorphism of the symmetric group $\mathfrak{S}_3$, i.e., the group of all permutations of 3 letters, is an inner automorphism.

2. Prove that the number of elements in any finite field must be a prime power.

3. Demonstrate that for any field K and any positive integer n the ring $\mathrm{M}_n(K)$ of $n \times n$ matrices with entries in K is a simple ring, i.e., a ring with no non-trivial proper two-sided ideal.

4. Find the elementary divisors of the finite abelian group
$$(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \quad .$$

5. Let M be a 3×3 matrix over the rational field $\mathbb{Q}$ whose characteristic polynomial is
$$t^3 - 2t^2 + t - 2 \quad .$$
Find all possible sequence of invariant factors for M.

6. Give a list of groups of order 35 that forms a complete set of representatives for the isomorphism classes of such groups.

7. Find the Galois group over the rational field $\mathbb{Q}$ of the polynomial
$$t^6 + t^5 + t^4 + t^3 + t^2 + t + 1 \quad .$$

8. Prove or disprove the following proposition: If A is a domain (a commutative ring without zero divisors) over which every cyclic (i.e., singly-generated) module is projective, then A must be a field.