Ph.D. Preliminary Examination in Algebra

August 31, 2001

1. Let A denote the matrix
$$\begin{pmatrix} 1 & -2 & 1 \\ 5 & -4 & 3 \\ 3 & -3 & 2 \end{pmatrix},$$
and let f be the $\mathbf{Q}$-linear endomorphism of the vector space $\mathbf{Q}^3$ given by $f(x) = Ax$. Find the dimension of the quotient vector space $\mathbf{Q}^3/\mathrm{Image}(f)$.

2. Let N be a normal subgroup of a group G with finite index $[G : N] = k$. Show that $g^k \in N$ for each element $g \in G$.

3. Why must the number of elements in a finite field always be the power of some prime?

4. Does the existence of the relationship
$$\left(2+\sqrt{-5}\right)\left(2-\sqrt{-5}\right) = (3)\,(3)$$
bear on the question of whether or not the ring
$$\mathbf{Z}[t]/(t^2+5)\mathbf{Z}[t]$$
is a principal ideal domain? (In this $\mathbf{Z}$ denotes the ring of integers, and $\mathbf{Z}[t]$ denotes the ring of polynomials in one variable over $\mathbf{Z}$.) Explain your answer.

5. Let M be the 4×4 matrix
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$
Find the characteristic and minimal polynomials of M when it is regarded as a matrix over the field $\mathbf{C}$ of complex numbers.

6. What is the Galois group of the polynomial $x^4 + 1$ over the field $\mathbf{Q}$ of rational numbers?

7. Prove over *any commutative ring* (with 1) that two isomorphic free modules of finite rank must have the same rank.

8. Find the group of all automorphisms of the symmetric group on 3 letters.