ALGEBRA PRELIMINARY EXAM

JANUARY 18, 2000

1. Show that a Sylow p-subgroup of D_{2n}, the dihedral group of order $2n$, is cyclic and normal for every odd prime p.

2. Show that every automorphism of S_3, the symmetric group of order 6, is inner, that is, is conjugation by an element of S_3.

3. Determine all abelian groups of order 144 up to isomorphism.

4. Let $f(x) = x^8 - 16$. Find the Galois group of the splitting field of $f(x)$ over the field of :

   (a) rational numbers ;

   (b) real numbers ;

   (c) integers modulo 17.

5. Find two non isomorphic rings of 9 elements whose additive groups are isomorphic. You must show that these rings are not isomorphic and that their additive groups are isomorphic.

6. (a) Let A be a commutative ring. Prove that every maximal ideal of A is a prime ideal of A.

   (b) Show that the ideal $(3, X)$ of $\mathbf{Z}[X]$ generated by 3 and X is a maximal ideal of $\mathbf{Z}[X]$.

   (c) Find a prime ideal of $\mathbf{Z}[X]$ that is not maximal.

7. Let $\mathbf{Z}[i]$ denote the ring of Gaussian integers. Prove or disprove: $\mathbf{Z}[i] \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{C}$ as rings.

8. (a) Prove or disprove : The $\mathbf{Z}$-module $\mathbf{Z}/2\mathbf{Z}$ is projective.

   (b) Prove or disprove : The $\mathbf{Z}$-module $\mathbf{Z}$ is flat.

   (c) Prove or disprove : The $\mathbf{Z}$-module $\mathbf{Q}$ is injective.