Ph.D. Preliminary Examination in Algebra

June 4, 1999

1. Let A be an $n \times n$ matrix with entries in the field $\mathbf{C}$ of complex numbers that satisfies the relation $A^2 = A$. Show that A is similar to a diagonal matrix which has only 0's and 1's along the diagonal.

2. Furnish examples of the following:
   (a) A finite group that is solvable but not abelian.
   (b) A finite group whose center is a proper subgroup of order 2.
   (c) A nested sequence of finite groups G, H, K with H a normal subgroup of G and K a normal subgroup of H such that K is not a normal subgroup of G.

3. Let p be the polynomial $p(t) = t^5 + t^2 + 1$ regarded as an element of the ring $A = \mathbf{F}_2[t]$ of polynomials with coefficients in the field $\mathbf{F}_2$ of two elements. Show that p is irreducible, and then find a polynomial of degree at most 4 with the property that its residue class modulo the ideal pA generates the entire multiplicative group of units in the quotient ring A/pA.

4. Let G be a finite group of order N, and let n be a positive integer that divides N. Do **one** of the following:
   (a) Prove that if G is abelian, then G contains a subgroup of order n.
   (b) Find an example of G, N, n as above where G has no subgroup of order n.

5. Show that every group of order 30 contains a normal cyclic subgroup of order 15.

6. Let F be the field $\mathbf{Q}(i)$ where $i = \sqrt{-1} \in \mathbf{C}$, and let E be the splitting field over F of the polynomial $f(t) = t^4 - 5$. Find:
   (a) the extension degree $[E : F]$.
   (b) the group $\mathrm{Aut}_F(E)$ of all automorphisms of E that fix F.

7. Let $\mathbf{F}_2$ be the field of 2 elements, and let R be the commutative ring
$$R = \mathbf{F}_2[t]/t^3\mathbf{F}_2[t].$$
   (a) How many elements does R contain?
   (b) What is the characteristic of R?
   (c) Find all ring homomorphisms $R \to R$.

8. Let a, b, c, d be elements of a field F, let A, B, C, D be $n \times n$ matrices over F, and let
$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$
If $\lambda : F^2 \to F^2$ and $\Lambda : F^{2n} \to F^{2n}$ denote the linear endomorphisms corresponding (relative to standard coordinates) to m and M, respectively, then to what linear endomorphism that may be constructed from λ and Λ may one relate the $4n \times 4n$ (Kronecker product) matrix
$$\begin{pmatrix} aA & bA & aB & bB \\ cA & dA & cB & dB \\ aC & bC & aD & bD \\ cC & dC & cD & dD \end{pmatrix} ?$$
Explain your answer.