

# Preliminary Examination in Algebra

Department of Mathematics & Statistics

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**Directions:** There are 8 questions, all of the same weight. Please take the time to ensure accuracy and completeness, especially for the questions you find easiest. (Completeness does not mean excessive verbosity. You should not attempt to prove standard propositions that you cite except where the proof of a standard proposition is explicitly sought.)

The ring of integers will be denoted by  $\mathbb{Z}$  and its field of fractions by  $\mathbb{Q}$ .

1. Prove that a non-abelian group of order  $2p$ ,  $p$  an odd prime, must have a trivial center.
2. When  $F$  is a field, let  $\text{GL}_n(F)$  denote the group of all invertible  $n \times n$  matrices in  $F$  under the operation of matrix multiplication, and let  $\text{SL}_n(F)$  denote its subgroup defined by restricting to matrices of determinant 1. Find a subgroup  $H$  of  $\text{GL}_n(F)$  such that  $\text{GL}_n(F)$  is isomorphic to the semi-direct product of  $H$  with  $\text{SL}_n(F)$ .
3. Prove that the number of elements in any finite field must be a prime power.
4. Let  $\mathbb{Z}/m\mathbb{Z}$  denote the ring of integers modulo  $m$ . Let  $r, s$  be positive integers.
  - (a) What element of  $\mathbb{Z}$  generates the ideal  $r\mathbb{Z} + s\mathbb{Z}$ ?
  - (b) What is the kernel of the canonical ring homomorphism  $\mathbb{Z}/rs\mathbb{Z} \rightarrow \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/s\mathbb{Z}$ ?
  - (c) Find an integer  $t$  such that  $\mathbb{Z}/r\mathbb{Z} \otimes \mathbb{Z}/s\mathbb{Z} \cong \mathbb{Z}/t\mathbb{Z}$ .
5. Let  $M$  be a  $3 \times 3$  matrix over the rational field  $\mathbb{Q}$  whose characteristic polynomial is

$$t^3 + 2t^2 - 4t - 8 \quad .$$

Find:

- (a) all possible sequences of (polynomial) invariant factors for  $M$ .
  - (b) representatives of the different possible similarity classes of such matrices  $M$ .
6. For any integer  $n \geq 3$  let  $D_n$  denote the  $n$ th dihedral group, i.e., the group of order  $2n$  that is the semi-direct product of the cyclic group  $\mathbb{Z}/n\mathbb{Z}$  with  $\mathbb{Z}/2\mathbb{Z}$  for the unique non-trivial action (by automorphisms) of the latter on the former, or, equivalently, the group of symmetries of a regular  $n$ -gon.
    - (a) Describe  $D_n$  by generators and relations.
    - (b) Show that every automorphism of the dihedral group  $D_3$  is inner, i.e., is the conjugation by some element of  $D_3$ .
    - (c) Show that for any  $n$  odd,  $n \geq 5$ , the dihedral group  $D_n$  has an automorphism that is not inner.
  7. For any integer  $n > 1$ , explain how to find a field  $K$  and a polynomial  $f(x) \in K[x]$  of degree  $n$  so that  $L = K[x]/(f(x))$  is a Galois field extension of  $K$  with cyclic Galois group of order  $n$ .
  8. Let  $\mathbb{F}_p$  denote the field  $\mathbb{Z}/p\mathbb{Z}$  of  $p$  elements. In  $M_n(\mathbb{F}_p)$ , the ring of  $n \times n$  matrices with entries in  $\mathbb{F}_p$ , let  $N$  be the  $n \times n$  Jordan block matrix with 1's on the first superdiagonal (and 0's everywhere else including the main diagonal). Let  $\mathbb{F}_p[N]$  be the  $\mathbb{F}_p$ -algebra of polynomials with coefficients in  $\mathbb{F}_p$  evaluated at  $N$ , let  $U$  be the group of units in  $\mathbb{F}_p[N]$ .
    - (a) Describe the elements of  $U$ .
    - (b) Find the exponent of  $U$  when  $p > n$ .
    - (c) Describe the isomorphism type of  $U$  as a finite abelian group when  $p > n$ .